RANDOM PROCESSES IN SYSTEM DYNAMICS

Erik Mosekilde and Steen Rasmussen, Physics Laboratory III, The Technical University of Denmark, 2800 Lyngby, Denmark.

Erik Mosekilde *is a senior Lecturer at the Technical University of Denmark where he teaches courses in modern physics, energy planning, and system dynamics. He is doing research in dynamic modeling of economic, technical, and physiological systems, and he is presently co-authoring a textbook on System Dynamics.*

Steen Rasmussen, a student at the Technical University, has just completed his M.E. thesis on "Structure, Fluctuations, and Stability".

ABSTRACT

Using examples from physics and operational research it is shown how DYNAMO can be applied to systems for which the development in time depends upon a series of random events. It is further discussed how the notion of a Markov process, fundamental to stochastic systems analysis, completely concurs with the idea of a state determined system underlying System Dynamics. Non-Markovian systems can usually be brought into Markovian form through a redefinition of state space, and we therefore conclude that System Dynamics has a general applicability to stochastic systems.

1. INTRODUCTION

The feed-back approach to management systems analysis developed in Industrial Dynamics[1] emphasizes the general structure of decision making rather than the individual decision. Management is to be considered as a continuous generation of control signals that adjust the flows of resources and materials through a system, with accumulation or depletion along the material flow lines gradually changing the conditions under which new decisions have to be made. As a reflection of this picture, the User's Manual[2] presents DYNAMO as a computer language which compiles and executes continuous simulation models.

Not all systems process a continuous flow of materials, however,[3,4] and sometimes a limited number of decisions can be of vital significance to the future development of a system. Moreoever, the available information is not always sufficient to determine the time at which a large material element passes from one state to another, or to determine the outcome of certain crucial decisions[5]. Under these conditions a discrete, stochastic description seems more appropriate than a continuous one. Industrial Dynamics has now developed into a general theory of dynamic systems with applications in a wide range of different disciplines, and the question therefore arises whether System Dynamics can be applied to stochastic systems, i.e., to systems for which the development depends on the outcome of a series of random events. We have already discussed this question in a communication to the System Dynamics Conference in Bruxelles[6]. It is the purpose of the present paper to investigate the problem in further detail and to show that System Dynamics and DYNAMO can be used quite generally to simulate stochastic systems. Compared with other methods used in this field, System Dynamics has the advantages of a well-developed flow-diagramming technique, a formalism which emphasizes feed-back and non-linearities, and a simple and efficient simulation language. The literature offers little evidence of applications of System Dynamics to stochastic systems, however, and for this reason we assume that our discussion can contribute to a clarification of the theoretical foundation of System Dynamics, as well as to a broader use of the method.

With the introduction of concepts such as dissipative structures and self-organization, Prigogine and his co-workers[7-10] have stimulated a rapidly growing interest in stochastic processes. Together with the works of Lorenz[11], May[12], Feigenbaum[13], and Mandelbrot[14] on bifurcation theory, chaotic behaviour, strange attractors, and fractal dimensions, the results obtained by the Bruxelles school have initiated the study of several new classes of instability phenomena. It is a promising area for future research to try to investigate some of these instabilities by means of System Dynamics. The contention that macroscopic systems can be extremely sensitive to small fluctuations[7] appears to contradict the assumption made by most system dynamicists that social systems are relatively insensitive to parameter variations. In our opinion, the conclusion which should be drawn from this difference in view is that system dynamicists must pay more attention to stability problems, to the problems associated with aggregation, and to the significance of random processes.

2. MONTE-CARLO SIMULATION

Diffusion is a characteristic example of a random process in physics. Although the molecules of a gas move in accordance with Newton's laws, their number is generally so large that it is impossible to imagine that we could follow the path of the individual particle in detail. On a macroscopic, phenomenological level, the process can be treated by means of a diffusion equation, i.e., a relation that specifies the development in time of the average density of a large number of molecules. For a more detailed treatment, however, we need a method that can perform the necessary transformation from the microscopic description to the macroscopic level.

An approach to this "transformation", which we shall refer to as a Monte Carlo simulation, is to follow the course of a single particle through a large number of scattering processes, considering each of them as a random event[15]. The outcomes of the individual scattering process (the time of flight since the last collision, the scattering angle, the energy loss, etc.), are then assigned according to certain probabilities derived from basic physical as well as from statistical considerations. Using this approach one can avoid all the approximations usually required to reach a solution in an analytical model. The disadvantage is that a very large number of scattering processes (10,000 or more) may have to be

considered to get satisfactory results. In many cases, however, one can acquire an impression of the characteristic behaviour of a system through a limited number of relative short simulations.

2.1 One – Dimensional Random Walk

As a first example of how DYNAMO can be used to perform Monte Carlo simulations, consider a one-dimensional random walk. A particle is bound to move along an x-axis with a constant speed VO. Each third time unit the particle undergoes a virtual scattering process in which, with equal probabilities, it either continues undisturbed or reverses its direction of flight. Formulated in DYNAMO, this problem may read:

```
L  X.K = X.J + (DT) (V.JK)
N  X = 0
R  V.KL = VO*CLIP (1,-1, SNN.K,0)
C  VO = 1
A  SNN.K = SAMPLE (NOISE ( ) ,3,.2)
```

Here, the state variable X and the rate variable V denote the position and the velocity of the particle, respectively. Each third time unit the auxiliary variable SNN samples the series of random numbers generated by NOISE( ). The CLIP function hereafter determines in which direction the particle moves. If SNN>0, then V>0 and the particle moves in the positive x-direction for the next three time units. On the other hand, if SNN<0, then V<0 and the particle moves backwards. With its constant intercollision time, the above description can be classified as a discrete time formulation. The one-dimensional random walk is usually formulated in a slightly different way by assuming that the time of flight between two collisions is exponentially distributed with an average time of flight TAU. The scattering probability is then independent of the previous history of the particle, and the probability that a scattering will occur in the time element DT (<<TAU) is DT/TAU. The particle is now assumed to reverse its direction in each collision.

A System Dynamics flow diagram of this system is shown in Figure 1. A particle bound to move along a single axis has two degrees of fredom, and we therefore need two state variables. Normally these would be the position and the momentum (or the velocity) of the particle. If we assume that the speed remains constant VO, we can also use the position X and the direction of flight DOF. The direction of flight is a discrete variable which equals +1 when the particle travels in the positive x-direction and -1 when the particle travels in the negative direction. The corresponding rate variable CID represents the change in direction.

It is worth noticing that the introduction of stochastic processes does not affect the structure of the flow diagram. The rate variables now describe either continuous processes (V) or abrupt transitions from one state to another (CID), and some of the auxiliary variables describe transition probabilities. However, the general rules of flow diagramming remain unchanged.

As the DYNAMO formulation of the one-dimensional random walk in the continuous time formulation we have used:

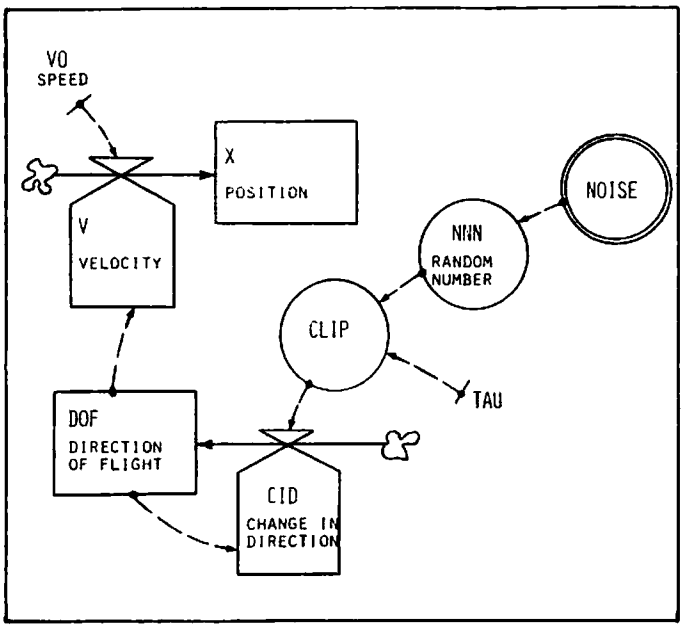

Figure1: Flow diagram for a one-dimensional random walk in a continuous time formulation. Due to the simplicity of the problem there is only one feedback loop, but the lack of significant feedback is not necessarily a characteristic of stochastic systems.

```
L    X.K = X.J + (DT) (V.JK)
N    X = 0
R    V.KL = VO*DOF.K
C    VO = 1
L    DOF. K = DOF. J + (DT) (CID. JK)
N    DOF = 1
R    CID. KL =- 2*DOF.K*CLIP(1/DT,0,DT/TAU,
     NNN.K)
C    TAU = 3
A    NNN.K = NOISE ( ) + .5
SPEC DT = 1
RUN
NOISE 345678
RUN
NOISE 234565
RUN
NOISE 567487
RUN
```

NNN.K generates a series of random numbers uniformly distributed between 0 and 1. By means of the CLIP function, NNN is compared with the probability DT/TAU that a scattering will occur in DT. If NNN < DT/TAU, this is taken

to mean that a scattering does occur, and the direction of flight is reversed. if NNN > DT/TAU, no collision is assumed to take place, and the particle continues undisturbed. In principle, the formulation is independent of DT. The simulation results will depend on DT, however, due to the effect that DT has upon the series of random numbers generated by NOISE and due to the fact that these random numbers are compared with a DT-dependent scattering probability.

Figure 2 shows a typical simulation result for DT = 1. (This means that the condition DT/TAU<<1 has not been strictly fulfilled). We can see how the particle is bounced back and forth. In the long run, the particle is likely to move away from its initial position, although of course there is no preferred direction for this walk-way.

Figure 3 shows a set of simulation results with a somewhat longer time scale and with four different initiations of the NOISE function. Note how such a set of simulations can be performed in RERUN mode by introducing additional NOISE cards.

2.2 Two – Dimensional Random Walk

The method described in the preceding paragraph can easily be extended to treat higher dimensional problems. If the speed of the particle is assumed to remain constant, we need

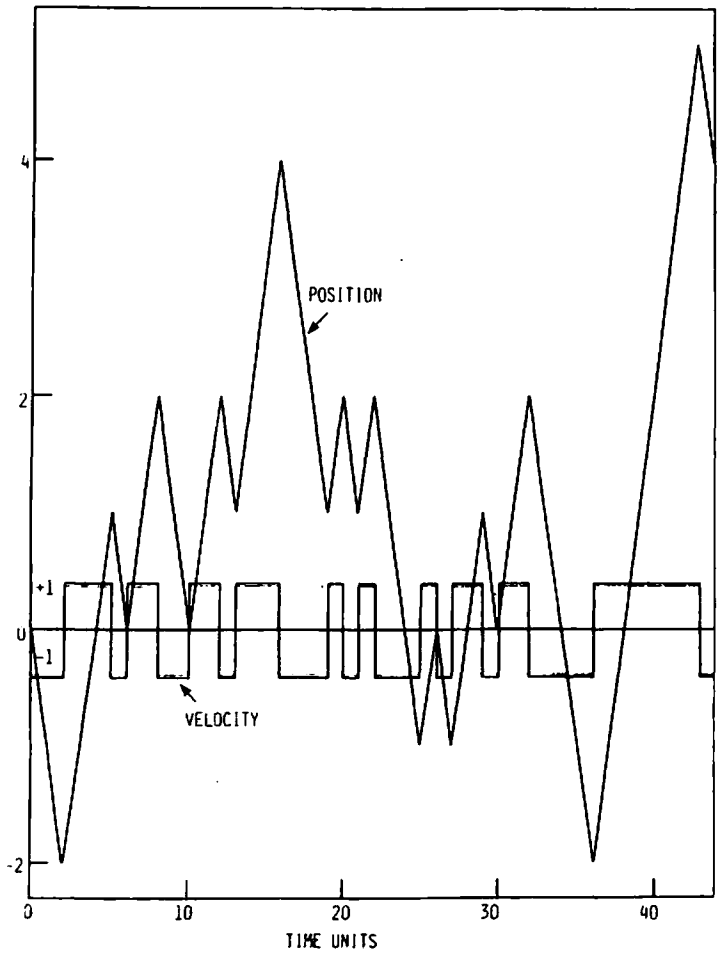

Figure 2: Simulation results for a one-dimensional random walk with exponentially distributed times of flight.

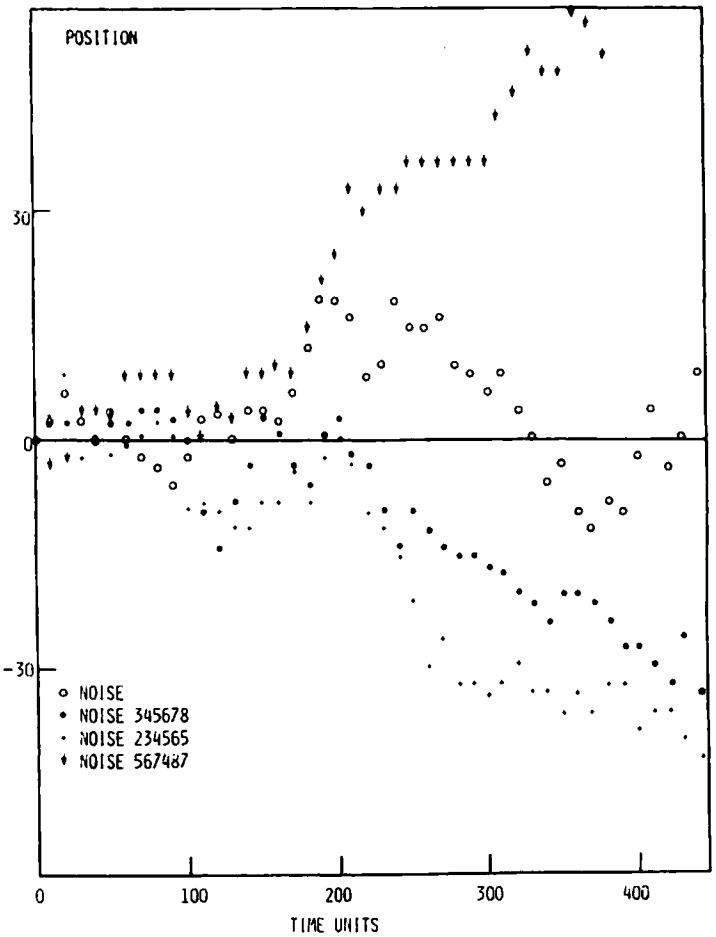

Figure 3: Simulation results for a one-dimensional random walk with four different initiations of the NOISE functions. Depending on the series of random numbers generated by NOISE, the particle can walk away from its initial position in either direction.

three state variables to describe a two-dimensional random walk. These can be the x-coordinate, the y-coordinate, and the angle ϑ between the direction of flight and the x-axis. The velocity components are determined by $V_x = V_0 \cos \vartheta$ and $V_y = V_0 \sin \vartheta$, and the only DYNAMO equation that deserves attention is the rate equation which determines the change in ϑ. For this equation we have used.

R CIT.KL = 3.141593*(NOISE( )+.5)*CLIP(1/DT,0,DT/TAU,NNN.K)

As before, NNN.K generates a series of random numbers uniformly distributed between 0 and 1. The CLIP function determines whether a scattering process occurs during the time element DT. The factor 3.141593*(NOISE( )+.5) adds a random fraction of π to ϑ in each scattering process. Note that we need two separate NOISE calls to secure that all scattering angles can be realized. If NOISE ( )+.5 in the equation for CIT is replaced by NNN.K, only scattering angles smaller than π (DT/TAU) will be realized, since only random numbers NNN < DT/TAU lead to scattering. At the same time, the formulation secures that the scattering angle is statistically independent of the time of flight since the last collision.

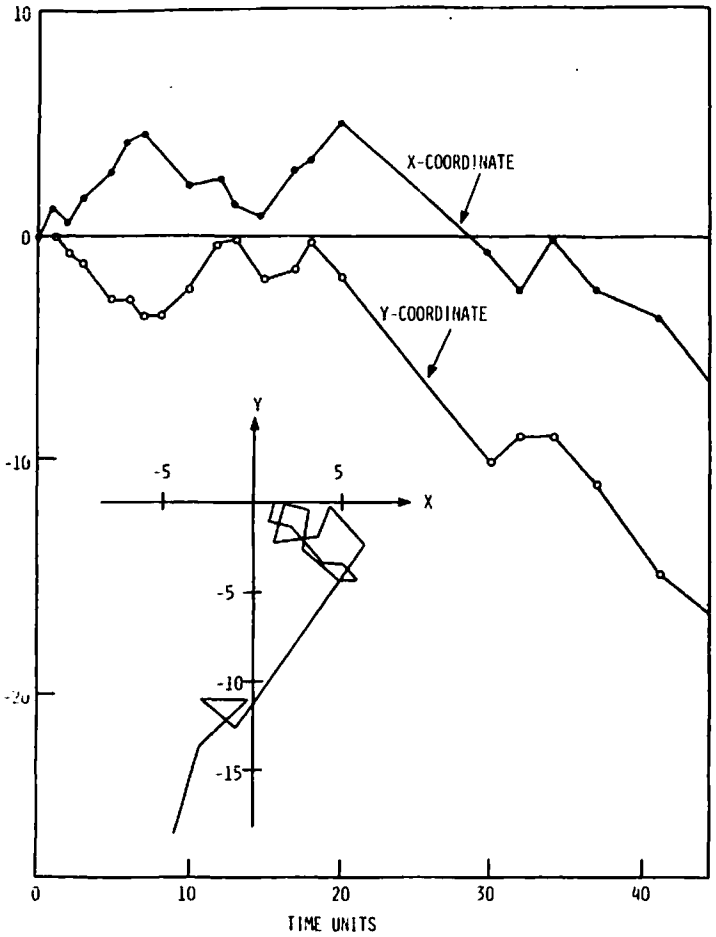

Figure 4: Simulation results for two-dimensional random walk with exponentially distributed times of flight, and with all scattering angles equally probable.

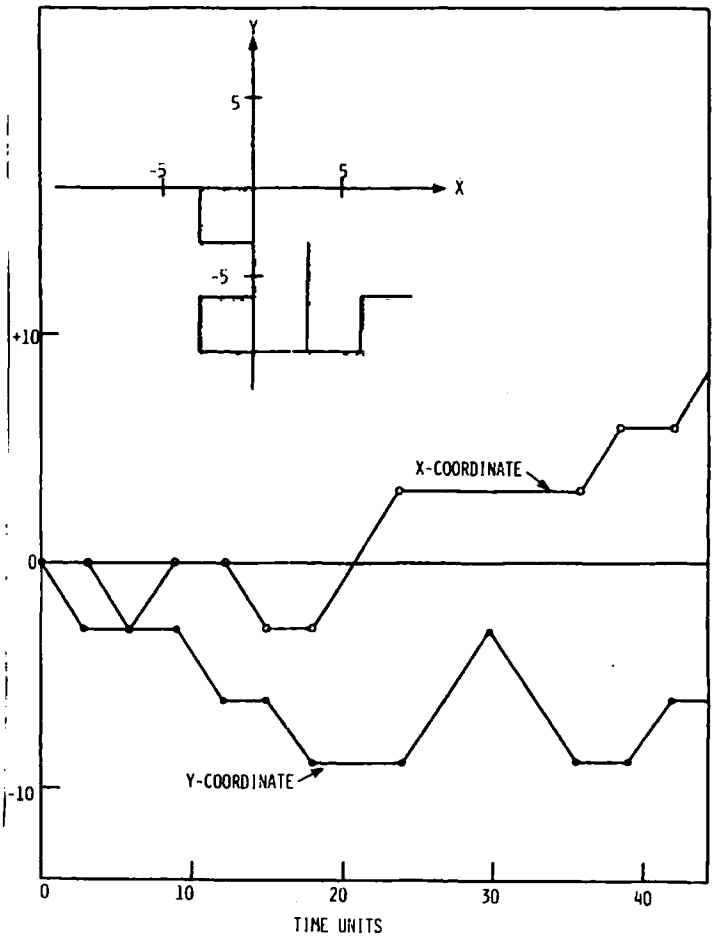

Figure 5: Simulation results for two-dimensional diffusion with constant time of flight, and with only four directions allowed.

Figure 4 shows a typical simulation result obtained with this model. The PHASE plot has been constructed manually from the development in the two-position coordinates. Figure 5 shows a similar result obtained under the assumption that there is a fixed intercollision time, and that the particle can travel only in one of the four directions specified by the coordinate system. This (discrete time and discrete space) model is of interest in connection with a discussion of the so-called self-avoiding random walk (see the paragraph on non-Markovian processes).

3. QUEUEING PROCESSES

An important field of application for the theory of stochastic processes is the study of queueing phenomena in communication, transportation, production, and service systems[16]. Consider a telephone switchboard with a certain number of outgoing lines. Since the operator usually has no information that allows him or her to know in advance when a particular customer wants to call, and since the various customers usually use the phone independently, calls to the switchboard can be considered as random events occuring singly in time.

Let us assume the existence of a parameter AROC, the average rate of calls, such that the probability that a call will occur during the time element DT is AROC*DT (<<1). Let us further assume that the duration of a call DOC is a random variable which is exponentially distributed according to

$$P\left\langle DOC>t \right\rangle = \exp\left\langle -t/ALOC \right\rangle .$$

Here, ALOC denotes the average length of a call. $P\left\langle DOC>t \right\rangle$ is the probability that the duration of a particular call will be longer than t. With this distribution of DOC, the probability that a particular call will be terminated in the time element DT equals DT/ALOC, independent of the previous history of that call.

As we shall see in connection with our discussion of the so-called Markov condition, this is a very crucial assumption. At the same time we shall indicate how one can do away with the assumption and treat problems with other kinds of distributions. For the moment, it suffices to note that the exponential distribution appears to be a reasonably good approximation for the duration of local telephone calls[16].

At any given moment, the state of the switchboard may be described in terms of the number of occupied lines NOC. Note that we are now using a macroscopic systems formulation in which the state variables measure the *number of elements* in the various possible microscopic states. The corresponding rate variables are the rate of incoming calls RIC and the rate of termination of calls RTC. The DYNAMO program may read:

L NOL.K = NOL.J + (DT) (RIC.JK-RTC.JK)

N NOL = 1

R RIC.KL = CLIP (1/DT,0,AROC*DT,NOISE( )+.5)

C AROC = .5 (calls per minute)

R RTC.KL = NOL.K*CLIP (1/DT,0,DT/ALOC, NOISE ( ) + .5)

C ALOC = 4 (minutes per call)

DT must be chosen such that both of the conditions AROC *DT<<1 and DT/ALOC<<1 are satisfied. In practice, this means that we should take DT≤ .2.

Figure 6 shows a typical simulation result with the above model. For more complicated systems, simulations of this kind may be used, for example, to determine the required number of lines.

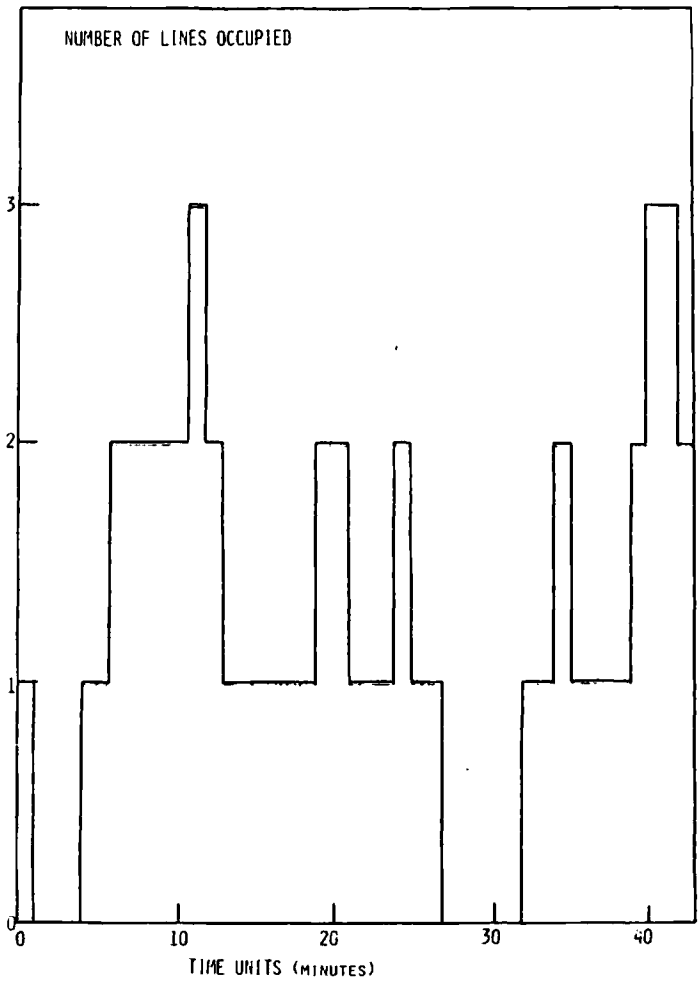

Figure 6: Simulation results for the simple switchboard model showing the number of occupied lines as a function of time with an average of .5 calls per minute and a mean duration of 4 minutes per call. In the present simulation, the maximum number of occupied lines is 3, but to some extent this number will vary with the set of random numbers generated by NOISE ( ).

The switchboard model can be "reduced" into a continuous, deterministic model through the replacements:

$$\text{CLIP}(1/DT,0,AROC*DT,\text{NOISE}(\ )+.5) \longrightarrow AROC$$

and

$$\text{CLIP}(1/DT,0,DT/ALOC,\text{NOISE}(\ )+.5 \longrightarrow 1/ALOC$$

The reason for using a stochastic description is that the number of calls handled at any given time is small, and that random fluctuations therefore play a significant role in the performance of the system. For a macroscopic system with a large number of elements one might expect that such an "aggregation of events" into continuous flows would generally be justified[2]. In the presence of instabilities, this aggregation procedure may become questionable, however, and the system behaviour may be controlled by small irregularities[5,7].

4. THE MARKOV CHAIN

Another stochastic problem from the field of operational research is the so-called Markov chain[16]. Consider a component such as a car battery which is subject to wear, and suppose that the batteries of a limited number of cars are inspected regularly each third month. By this inspection the batteries are classified as being in one of three states:

state 1: good

state 2: average

state 3: poor

Average means that although the battery still functions, the specific weight of the electrolyte has started to change, and the recharging time has become a little too long. Poor means that the condition of the battery is now so unsatisfactory that it is likely to fail before the next inspection.

Figure 7 illustrates how the structure of this problem is usually presented. The P_{ik} parameters here denote the transition probabilities between the various states. Thus, P_{12} is the probability that, assuming a battery is found to be good in one inspection, it will be only average in the next. P_{11} is the probability that a battery remains good from one in-

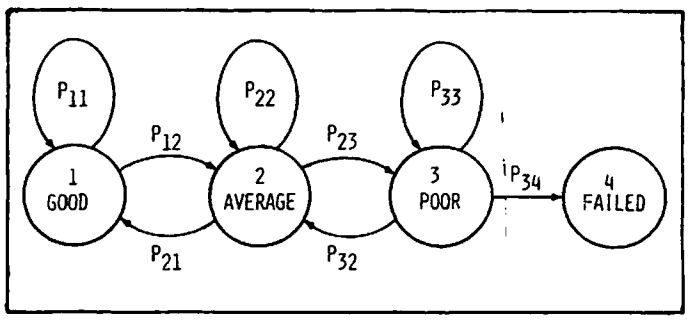

Figure 7: A Markov chain describing the development of a (limited) number of car batteries as they pass from state 1 (good) through states 2 (average) and 3 (poor) to final failure. Such a model can be used for instance to devise a proper policy for maintaining the batteries.

spection to the next, and P_{21} is the probability that a battery recovers from average to good between two inspections. Similar definitions hold for P_{22}, P_{23}, P_{32}, P_{33}, and P_{34}.

To simulate this problem with DYNAMO, we need three subsequent state variables measuring the number of batteries in each of the three microstates and connected with rate variables corresponding to the various possible interstate transitions. To generate these transitions stochastically, we need three independent NOISE calls, but otherwise the problem is straightforward. Our purpose in discussing it here is to make comparisons in the discussion of non-Markovian processes.

5. THE MARKOV CONDITION

Stochastic processes may be divided into discrete time and continuous time processes. The one- and two-dimensional random walks with constant time of flight, and the four-state Markov chair for automobile batteries are examples of discrete time processes, while the one- and two-dimensional random walk processes with exponentially distributed times of flight and the telephone switchboard are examples of continuous time processes.

In the processes that we have considered so far, the transition probabilities at a given time (or, for discrete time systems, in a given time interval) only depended on the state occupied at that time, and on the possible final states. It did not depend on what had happened before that time. This is known as the Markov condition[16], and systems satisfying this condition are referred to as Markov systems. Since for a Markov system the future path of any particular element does not depend upon the previous history of that element but only on its present state, we do not need to follow the passage of the individual element from state to state but we can lump all elements which are in the same microstate together and use the number of elements in each microstate as state variables. The corresponding rate variables will then be determined by the state variables, and the Markov condition is therefore equivalent to the assumption of a state-determined system underlying System Dynamics. We may thus conclude that System Dynamics can be applied to all stochastic systems that satisfy the Markov condition, and for which time is the only independent variable.

6. NON-MARKOVIAN PROCESSES

Not all stochastic processes satisfy the Markov condition, however, and it is therefore of interest to find out to what extent System Dynamics can be applied to non-Markovian systems. As a simple example of a non-Markovian process, consider a modified version of the wearing process for automobile batteries. Suppose that when a battery has entered state 2 (average), it either recovers before the next inspection or it remains average. Suppose further that if a battery is found to be average in two subsequent inspections, it always decays into state 3 (poor) before the following inspection. Since, with these assumptions, a battery spends two time periods in state 2 before decaying into state 3, the transition probability between states 2 and 3 depends on the time that a battery has already spent in state 2, i.e., upon the history of the battery. The Markov condition thus appears to be unsatisfied.

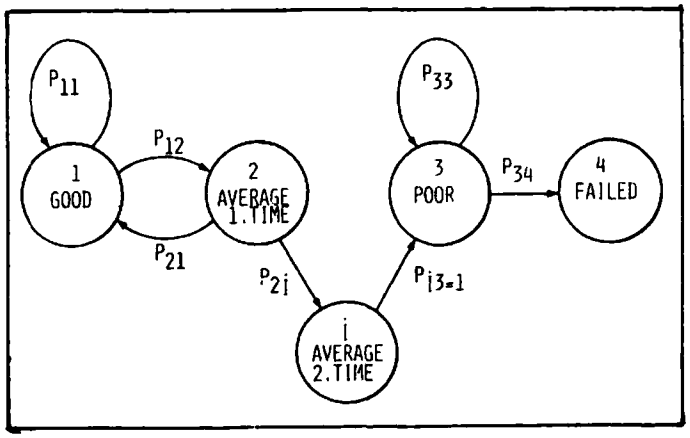

Figure 8: Conversion of a non-Markovian process into a state-determined process through the introduction of an intermediate state.

As illustrated in Figure 8, however, a simple extension of state space will convert the process into a Markov process. We just need to introduce an intermediate state, state i (average, 2. time) between state 2 (average, 1. time) and state 3 (poor). In System Dynamics this corresponds to the introduction of an extra level (or more generally extra levels) into the material flow line. We may therefore conclude that discrete-time problems for which the non-Markovian character is associated with the elements spending a well-defined time in certain states can be treated in System Dynamics.

The situation becomes a little more complex when we turn to a continuous-time problem. Consider again the telephone switchboard and assume that all calls last precisely 4 minutes. If we were to treat this problem rigorously by introducing inter-mediate states, we would need an infinite number of such states. An alternative method is to construct what we may call a System Dynamics stopwatch, i.e., a rate-level module which integrates TIME from the initiation of a call. When 4 minutes has passed, the stopwatch, by means of a logical function, terminates the call, and at the same time resets itself to be ready for the next call. This method requires a stopwatch for each call that the switchboard can handle at a time, and it is not possible to lump the calls together and consider the state of the system as determined by the number of occupied lines. Nontheless, we conclude that the problem is managable with System Dynamics.

In stochastic systems analysis[16] the term renewal processes is used to designate processes for which the time an element spends in a given state is described by a probability distribution different from the simple exponential distribution used in our first version of the switchboard model. The length of a telephone call, for example, could follow a distribution with a maximum corresponding to a most probable duration. Renewal processes can also be treated by means of the stopwatch concept.

As a final example of a non-Markovian process, consider the so-called self-avoiding random walk[16]. Suppose we have a two-dimensional diffusion process with constant time of flight and four possible directions of travel (see Figure 5), but assume that the problem is redefined in such a way that the particle can never visit the same place twice. This means that the system must always keep track of the history of the particle[17].

The way to treat this problem is to consider the background of regularly distributed points relative to which the particle moves as part of the system. With each collision point we can associate a state variable, and we can let the value of this state variable change (for example, from 0 to 1) when the particle passes the point. DYNAMO II is clearly not well-suited to manage the coordination problem associated with a simulation of this process. The important conclusion, however, is that even such a drastically non-Markovian process as the self-avoiding random walk can be turned into a state-determined process through a proper definition of state space.

7. CONCLUSION

The viability of System Dynamics certainly depends on the successful application of the method to significant real life problems. It is also important, however, that the basic assumptions of the method are continuously reconsidered and that the limits of applicability are clarified and, if possible, extended. One way of doing this is to test the method on different classes of relatively well understood problems, for instance, from the natural sciences. As an example of this we have considered the application of System Dynamics to a number of simple stochastic processes. Our analysis has shown that System Dynamics has a very general applicability to stochastic systems, and that DYNAMO is fully equipped to simulate most random processes. Some of the examples that we have discussed have not involved feedback loops. This is not a characteristic of stochastic processes, however, but a reflection of the simplicity of the system we have considered.

The question that now arises is: how can stochastic phenomena affect the behaviour of macroscopic social systems to which the "law of large numbers" should apply. As we have already stated, the answer to this is that in the presence of instabilities, random noise components can amplify over many decades until they suddently become strong enough to change the behaviour of the system dramatically. If the unemployment in increased, for instance, our Western societies may become unstable with respect to a social revolution in the sence that nationwide upheavals could be triggered by almost any outbreak of local unrest. In a decade or two, when less than half of the population has been vaccinated, the human race may be unstable with respect to the re-appearance of smallpox. This implies that a local outbreak of this disease could explode into a major epidemic wave which would propagate all around the world.

These examples may be considered as relaxation type transitions, which could occur even in isolated systems. The main idea of the Brussels school[7,8] is that human societies are open systems in which large amounts of energy and resources are processed. These flows of energy (useful energy) so to speak "lift" the systems far away from equilibrium and permit spontaneous "upward transitions" in which structure is created, and diversity and specialization increased. In a forthcoming paper[17] we shall discuss these problems in more detail and also show how System Dynamics can be used to model self-organizing processes.

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