# UNITY-GAIN POSITIVE FEEDBACK SYSTEMS

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**ABSTRACT**
Many feedback systems, when tested by applying exogenous inputs, are able to maintain their characteristic behavior in the face of moderate changes in their internal relations. The robustness of behavior is usually due to the presence of compensating feedback. One class of systems seems to show a much lower resistance to external disturbances. The behavior of systems of this type – unity-gain positive feedback systems – depends in detail on the exogenous influences that impinge on the levels. In particular, the equilibrium state that results from a given disturbance depends on the size and duration of the disturbance. In this paper we show that these systems have a constant of the motion which accounts for their unique behavior when subject to exogenous disturbances. As well, the constant of motion permits a clear definition of the polarity of a large class of feedback structures.

System dynamics studies devote much effort to understanding the relationships between structure and behavior. This effort is inspired by the working hypothesis that the structure of a system is the main determinant of its behavior over time – at least in general terms. Certainly, the details of behavior depend on the exact form and precise magnitudes of the operating relationships. But it is often observed that the behavior itself, as summarized in a verbal description, is relatively insensitive to detailed knowledge of the relationships. This hypothesis has served to permit the construction of models of systems in which the *precision* of parameter values is not as central to an effective understanding of system performance as in other modeling paradigms.

The validity of this working hypothesis is challenged by the existence of certain systems for which the behavior depends in detail on the time history of exogenous influences on it. In particular, such systems may have no unique equilibrium state independent of the initial conditions. A striking example is provided by the Salesman-Backlog loop in the classic 'Market Growth' paper.[1] For purposes of illustration, a simulation of this sub-structure, independent of the Capacity Expansion and Delivery Delay loops in the full model, is performed by fixing the Production Capacity and Salesman Effectiveness parameters to be constant. The Production Capacity is taken to be 12000 un/mo and the Salesman Effectiveness is held constant for a succession of 20 month periods at values of 400, 167, 100 and 167 un/man-mo. The equations necessary to replicate the simulation are found in Appendix A and the result is shown in Figure 1.

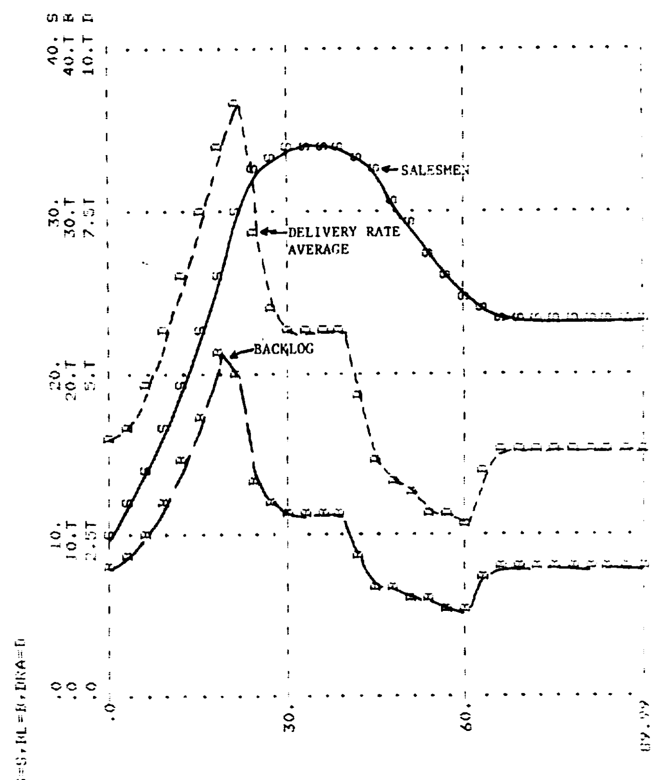


*Figure 1: The Salesman-Backlog Loop Simulated with Salesman Effectiveness Fixed for a Succession of 20-month Periods*

The simulation starts from initial conditions far from the 'equilibrium' determined by the upper limit on Delivery Rate and Orders Booked provided by the Production Capacity. The production capacity limit imposes an equilibrium on the delivery rate and the number of salesmen; when the rate of orders booked is larger than the delivery rate, the backlog continues to grow indefinitely so that the loop is not truly in equilibrium (all level variables constant). The Salesman, Backlog and Delivery Rate Average levels pass through a period of growth, adjustment to equilibrium, decline and adjustment to a different equilibrium respectively. The equilibria result from fixing the Salesman Effectiveness at 167 un/man-month, when the revenues generated by salesmen are just sufficient to support the number of salesmen employed. Figure 1 shows that the equilibrium depends on the 'initial' conditions, the state at the beginning of each 20-month period when the Salesman Effectiveness equals 167 un/man-month. In a state-determined system, there is nothing dynamically unique about an 'initial' time. The behavior of the system is the same regardless of how the 'initial' state is created. The non-unique equilibria displayed in Figure 1 can equally well arise from imposing two pulse inputs on one of the levels when the Salesman Effectiveness is 167. Any exogenous input can be expressed as a series of pulses. *This structure is sensitive to the detailed history of exogenous inputs impinging on it.* This sensitivity, or non-uniqueness of the equilibrium state, is not evident from the analysis presented in the original reference. In this paper, we look at a class of systems whose behavior over time, especially the equilibrium state or the approach to equilibrium, appears to depend on the detailed set of influences on the system rather than on the general structure.

## UNITY-GAIN MODELS

The class of systems in question was studied by Graham[2] who named them unity-gain, positive feedback systems. These systems are represented by models in which the links between levels are such that in tracing around the major feedback loop that defines the system, the resulting relationships *appear* to be deviation-enhancing (positive) rather than deviation-suppressing (negative) but their behavior is goal-seeking. In the 'Market Growth' example cited above, an increase in the number of salesmen leads to an increase in the orders booked and in the backlog, subsequently to an increase in the delivery rate and hence in the budget to hire more salesmen, and hence to an increase in the number of salesmen. The initial increase in salesmen leads to a further increase in salesmen. By this reasoning, we seem to have a positive loop. Yet in the two periods when the salesman effectiveness is set to 167, the levels tend towards non-trivial equilibrium values – they are goalseeking. The goal-seeking behavior, which is characteristic of first-order, negative loops, contradicts our expectations based on the designation of a positive loop. The use of causal loop diagrams to analyze behavior has been criticised and defended by several authors[2,3,4] but for these kinds of models, the value of causal loop analysis is difficult to establish on present evidence.

The behavior of this kind of feedback loop depends in part on the open-loop step gain experienced by a variable in the loop when a perturbation is applied to it exogenously. The open-loop step gain is defined as the relative change in the value of a level resulting from a step change in an input to the level. It is determined by opening the loop at an input rate to the level, introducing a step in the rate and observing the effect on an equivalent rate variable as time increases to infinity.[5] The result of this process is the same as starting from a level and tracing the effects of all the auxiliaries encountered once around the loop while ignoring any delays. A more detailed argument to support this assertion is supplied in Appendix B. A simple-minded analysis based on tracing the effect of a change in a variable around the loop would lead one to expect amplification because the loop is positive. In the example cited, the gain is determined by the ratio of parameters

SE*RS/SS

where

SE = Salesman Effectiveness (un/man-month), varied
RS = Revenue from Sales ($/un) = 12
SS = Salesman Salary ($/man-month) = 2000

Under conditions of growth (up to month 20), the gain is greater than 1 so that the number of salesmen and the orders booked grow. When the gain is artbitrarily reduced to below 1 (but still positive, as in the period from month 40 to 60), the system decays towards a goal of zero salesmen and orders. When the gain is exactly +1 (in the periods from month 20 to 40 and 60 to 90), the system adjusts to an equilibrium that depends on the state of the system at the time when the gain became equal to unity. The singular feature of *unity* open-loop step gain is the source of the *sensitivity* of behavior that is typical of the systems of interest in this paper.

An important and subtle characteristic of these models is the fact that they possess a *'constant of the motion'*, i.e. a linear combination or weighted sum of the level variables in the major loop that has its time derivative identically equal to zero. The possibility of such a constant is suggested by the goal-seeking behavior, and the relatively rapid adjustment to equilibrium evident in Figure 1 when the gain is unity. The implications of this feature are far-reaching as we shall see.

## CONSTANTS OF THE MOTION

Though often unremarked, the existence of constants of the motion is not unusual. One example, adapted from Mass[6] is the simple Job-Vacancies and Employment model shown in Figure 2. Without enquiring into the details of the auxiliary

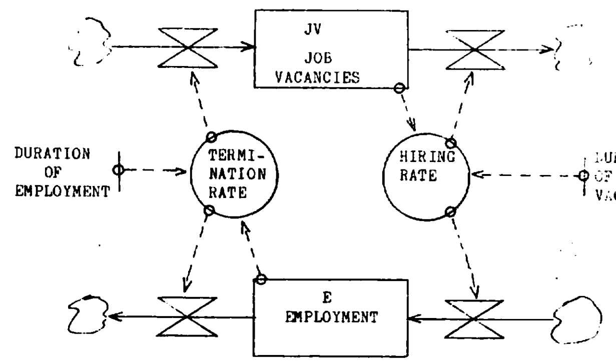


*Figure 2: Job-Vacancies and Employment Model*

functions, we can readily see that the rate of change of Job Vacancies is the net result of the Termination Rate – the firing of unsatisfactory employees increases the number of job openings, and the Hiring Rate – the hiring of new employees reduces the number of unfilled positions. The rate of change of Employment is the converse, being increased by the Hiring Rate and decreased by the Termination Rate. It is evident that

$$\dot{E} + \dot{JV} = (HR - TR) + (TR - HR) = 0$$

so that the combination (E + JV) is constant. In this case the constant of motion is easily interpreted as the total number of jobs available, both filled and unfilled. The total is constant in this model since no job creation or destruction mechanism is included. If there is an exogenous component in the Hiring Rate, for example a random input, the value of the sum is indeterminate and depends on the time series of exogenous shocks impinging on the system. In much the same way, the Salesman-Backlog simulation showed an indeterminacy in the equilibrium values reached during the two periods when Salesman Effectiveness was equal to 167. Depending on the values of the levels when the effectiveness was fixed so that the gain was +1, the directions and magnitudes of the adjustments varied from one 20-month period to another.

The constant of motion in the Job-Vacancy Employment model is something of an accounting identity and suggests that other similar structures may be found where system elements shuttle between a variety of levels while maintaining the number of elements constant except for inputs from sources exogenous to the positive loop under consideration. Material recycling models[7] and epidemic models[8] that share this characteristic are some indication of the wide applicability of these structures.

However, constants of the motion appear in other models that are not formulated as accounting identities. A simple example will be instructive before returning to the more complex and more general Salesman, Backlog, Delivery Rate Average structure. The example chosen was proposed by Graham[2] and describes a simple (not entirely realistic) model to determine the selling price (producer's price) of a product based on the direct costs of production and a margin to cover overhead and profit components. The DYNAMO diagram is shown in Figure 3.

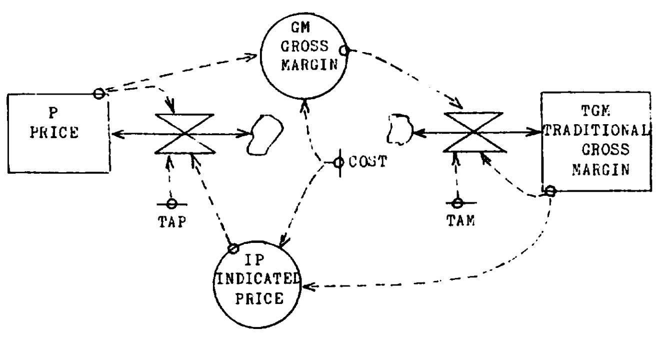


*Figure 3: Pricing Model – DYNAMO Diagrams*

The equations of the model are recorded in Appendix C. In this system, the price P at any time is used to determine the gross margin GM by subtracting the direct costs C, assumed to be constant for purposes of this example. The gross margin is smoothed to determine an average or 'traditional' gross margin TGM which is simply an exponential smoothing over a period TAM, the time to adjust the margin. The traditional margin is subsequently used to determine the desired or 'indicated' price IP by adding to the traditional margin the direct costs of production. The indicated price when compared to the current price is used to determine any change in price needed to bring the current price in line with the indicated price. The time period over which any change in price is effected is the time to adjust price TAP.

A little thought reveals that the model can easily be reformulated in terms of new level variables which have the same dimensions, namely gross margin GM and traditional gross margin TGM. The algebraic steps are sketched in Figure 4. The units of gross margin and price are both (\$/un).

$$P = \frac{IP-P}{TAP} = \frac{TGM+C-P}{TAP} = \frac{TGM-GM}{TAP} = \frac{(P-C)}{TAP} = GM$$

$$TGM = \frac{GM-TGM}{TAM} = \frac{-TGM+GM}{TAM}$$

*Figure 4: Pricing Model – Redefinition of Levels*

However, gross margin and traditional gross margin are concepts which are closer in meaning than are price and traditional gross margin. The re-definition of variables is inspired by a desire to develop a reduction in the number of concepts needed to understand the behavior. We note that a re-definition in terms of price and indicated price levels could equally well have been used.

The simple trick of rewriting the model in terms of variables which are dimensionally equivalent reveals that, after multiplication by TAP and TAM, the rates of change of GM and TGM are equal and of opposite sign. Thus the rate of change of the sum .

TAP*GM + TAM*TGM

is zero and the sum is a constant of the motion. If there are no exogenous influences on the levels, the sum remains constant. The value of the sum is given by the initial values of the levels GM and TGM. If GM equals TGM, both levels stay constant and the above sum is trivially constant. If GM is not equal to TGM initially, the two variables adjust to some compromise, equilibrium value while the above sum remains constant.

## DIMENSIONAL IDENTITY IN LOOPS

The fact that the *levels can be re-defined so that they all have the same dimensions* is a feature of all loops that is extremely important. Any level in a loop can be chosen to specify the dimensions of a standardized level variable for the loop. The other levels can be re-defined to be dimensionally equivalent by merely multiplying by appropriate conversion factors taken from the auxiliary relations in the loop. When the gain is exactly 1, any change in scale or dimensions introduced by a (possibly non-linear) auxiliary relation must be reversed or undone by subsequent auxiliaries in tracing once around the loop. *The net steady-state effect around the loop must be equivalent to an identity both dimensionally (as in any loop) and physically.*

## PHYSICAL IDENTITY IN UNITY-GAIN LOOPS

The physical identity is subtle and interesting. An open-loop step gain of +1 means that in tracing the change in the magnitude of a variable (induced by a prior change in that variable) once around the loop, the resultant change due to the auxiliaries is equal in magnitude to the original. *The addition of any number of units to a given level is simply preserved in the level.* This description ignores the transitory effects due to the delays as befits an equilibrium analysis. These effects play an important role in determining the final values of the different levels in equilibrium following an impulsive change in one of the levels in the loop but not the value of the constant of the motion.

Unity-gain loops are 'sensitive' to the effects of exogenous inputs since the levels in the loop will tend to preserve any changes impinging on them from outside the loop. Figure 1 demonstrates this sensitivity of unity-gain loops. Laplace transform theory shows that non-equilibrium initial values and impulse inputs are equivalent means for stimulating behavior modes. The difference in equilibrium values reached during the two periods when the gain is +1 is due to the difference in 'initial values' of the levels, at the instants when unity gain is imposed.

## A GENERIC STRUCTURE FOR UNITY-GAIN POSITIVE FEEDBACK LOOPS

The examples of constants of the motion that have been dealt with so far are either trivial (the Job-Vacancy-Employment accounting identity) or specialized (the Gross Margin-Traditional Margin case of linearly linked *information* delays). We return to the Salesman-Backlog model which has a more general structure of material and information delays linked by non-linear auxiliaries. Similar examples may be found as sub-models in the References.[9,10,11] These are sufficient to suggest that there is a generic structure, usually embedded in a larger model, and that an understanding of its characteristics is of some importance. The generic structure in question is illustrated by the operator-matrix equation shown in Figure 5. The equation is derived from re-writing the active DYNAMO equations in Appendix A in standard state-vector form. A slight modification of notation is made to accommodate the non-linear function PCF. To avoid using clumsy parentheses to indicate general functional dependence, a ring symbol (o) is introduced. In this example, the equations in Appendix A allow us to write

$$PCF(BL/12000) = PCFoBL$$

$$\begin{pmatrix} \dot{S} \\ \dot{BL} \\ \dot{DRA} \end{pmatrix} = \begin{pmatrix} -1/SAT & 0 & RS/(SS*SAT) \\ SEM & -PC*PCFo & 0 \\ 0 & PC*PCFo/DRAT & -1/DRAT \end{pmatrix} \begin{pmatrix} S \\ BL \\ DRA \end{pmatrix}$$

*Figure 5: Salesman-Backlog Model Operator-Matrix Form*

The equations show a structure composed of a linked set of three delays if we note that the non-linear function PCF is piece-wise linear in BL with positive slope. The off-diagonal terms which link one level to its predecessor in the loop are all positive so that the major loop is indeed positive. The generic structure, of which this is only an example, has:

(i) negative terms on the diagonal to represent an outflow from a level which depends on the amount in the level. This is a generalization of linear or exponential delays. The terms on the diagonal may be non-linear functions of the level subject to the condition that when the level is zero the function is zero also.

(ii) non-zero terms only below the major diagonal and in the 'north-east' corner to form the links between levels for the major loop. Alternative forms are found by changing the order of writing the levels, but these are clearly equivalent since the ordering of the levels is of no significance for the dynamics.

The reason why this structure can be considered *generic* is evident. Physical consistency requires a delay-type structure: if there is nothing in a level, the outflow must be zero. The links between levels are necessary to form the major loop. All other off-diagonal terms represent sub-major loops, of no interest to this study. The structure is a physically consistent, major loop of general form.

## CONSTANT OF MOTION – DERIVATION

A constant of the motion is a linear combination of the levels and corresponds to adding together multiples of the rows in an operator-matrix of the sort shown in Figure 5, so that the resulting sum is identically zero. The possibility of finding a constant of the motion in such a structure is suggested by the observation that each column contains only two entries, and they have opposite signs. Thus, some method of adding rows together could result in a right-hand side which is zero. A procedure to capitalize on this observation, when the equations are written in a form like Figure 5, is the following:

– Start with the last row, look for a multiplier for the first non-zero element such that it cancels the term in the same column in the row immediately above.

–Continue in the same way up the rows until you reach the first. Apply the multipliers found this way to the first row. If the loop is unity gain, the last term in the first row will cancel the last term in the last row.

For the Salesman-Backlog model, it is easy to see that the procedure leading to the combination

$$COM = DRAT*DRA + BL + SAT*SEM*S$$

gives the equation

$$\dot{COM} = (SEM*RS/SS - 1)DRA$$

Thus for the values of RS and SS given previously, we see that when

$$SEM = 166.6 \doteq 167$$

the rate of change of the above linear combination of the levels is indeed identically zero.

Alternatively, one can specify a linear transformation of the vector equation in Figure 5 that accomplishes the same result as the procedure described above. The transformation is simply to multiply the vector equation by the vector

$$(SEM, 1, 1) \begin{pmatrix} SAT & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & DRAT \end{pmatrix} = (SEM*SAT, 1, DRAT)$$

It is a very special feature of this model, not true in general, that the function PCF has no effect on the constant of motion. In general, the delay time for any level will appear in the constant of motion. The pricing model shows this feature. In the Salesman-Backlog model, the delay time for the backlog is the reciprocal of the slope of the function PC*PCFo. The delay time for the Backlog also appears in the link between Backlog and Delivery Rate Average as part of the major loop. This feature permits an exact cancellation of the PCFo terms in the rate of change of the constant of motion.

The derivation of the constant of the motion shows the unique nature of unity-gain, positive feedback loops. *The constant exists only when the positive major loop is exactly compensated by a minor negative loop.* If the product of the internal gain parameters SEM*RS/SS is not exactly equal to +1, the rate of change of the 'constant' is always proportional to one of the levels in the loop. The time derivative of the 'constant' is then either positive (gain>1) or negative (gain<1). The general form of the constant of the motion is developed in a subsequent paper,[12] which it is hoped will be published in a forth-coming issue of DYNAMICA.

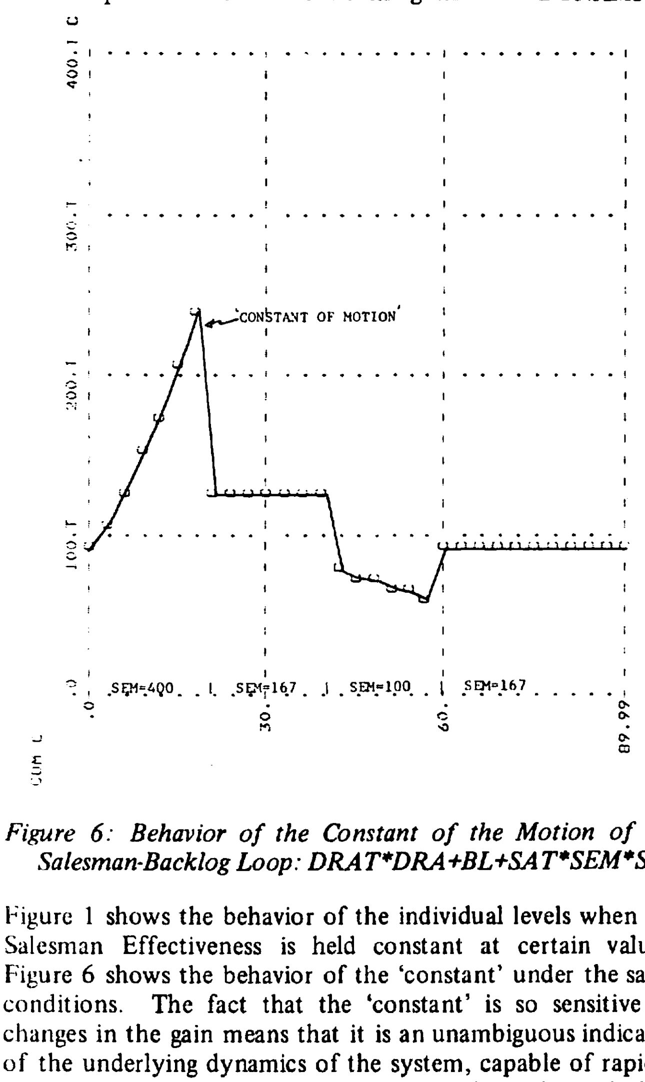


*Figure 6: Behavior of the Constant of the Motion of the Salesman-Backlog Loop: DRAT*DRA+BL+SAT*SEM*S*

Figure 1 shows the behavior of the individual levels when the Salesman Effectiveness is held constant at certain values. Figure 6 shows the behavior of the 'constant' under the same conditions. The fact that the 'constant' is so sensitive to changes in the gain means that it is an unambiguous indicator of the underlying dynamics of the system, capable of rapidly distinguishing growth, stagnation and decline phases before they are evident in all the levels.

## FURTHER OBSERVATIONS

### Equilibrium

Because each level in the system is in a delay structure, each level has a tendency to adjust to an equilibrium. The existence of a constant of the motion means that at least one equilibrium exists. This follows from the fact that the constant of the motion is trivially equal to its value at the initial time, and that the delay structure keeps all levels bounded in magnitude. The existence of a constant of the motion means that if a level is perturbed (or if it is not at its equilibrium value) all the other levels must adjust in such a way as to preserve the constant of the motion (whose time rate of change is zero). By acting to preserve the constant of the motion, a system may nevertheless show some apparent arbitrariness in its equilibrium state since different levels may compensate by different amounts and in different directions for the effects of exogenous pulse inputs (or equivalently, for the fact that some or all of the states are initially not at their equilibrium values). The adjustment process and the equilibrium are not in fact arbitrary. The response to a given step input is deterministic in the sense that if the model starts in the same initial condition it always approaches the same equilibrium. However, if it starts from differing initial conditions, the equilibria will differ. If the initial conditions are close to each other so also will be the equilibrium states. The pathological case of exactly compensating, unlimited increases and decreases in components of the constant of motion is seen to be impossible because the delay structure acts to return any runaway levels to equilibrium.

### Stability

An equilibrium of a unity-gain loop is stable but not asymptotically stable, i.e. the equilibria resulting from two initial conditions that are close to each other are also close to each other but do not tend to approach each other over time. In Figure 1, there is no tendency for each level to approach a common limit for that kind of level when the gain is unity. Equally well, if a model is started from the same initial conditions but is exposed to two slightly different streams of exogenous influences, the time paths of the two simulations will be approximately the same but these paths will show no tendency to converge on some unique path.

One can demonstrate that small differences in initial conditions give small differences in equilibria by simulating the Salesman-Backlog model with the same sequence of values of Salesman Effectiveness (400,167,100,167) applied over the periods (0,21), (21,41), (41,61), (61,90) and (0,19), (19,39), (39,59), (59,90). The results are shown in Table 1 where the final values, at TIME = 90, of each level and the constant of motion are recorded for each case.

| *First Interval* | *Backlog* | *Salesmen* | *Delivery Rate Average* | *Constant of Motion (/1000)* |
|---|---|---|---|---|
| (0,19) | 7433 | 22 | 3717 | 85.48 |
| (0,20) | 7827 | 23 | 3913 | 90.01 |
| (0,21) | 8240 | 25 | 4120 | 94.76 |

Table 1: Sensitivity of the Unity-gain Response of the Salesman-Backlog Model to Variations in the Initial Conditions.

### Random or Unknown Inputs

In the presence of random or unknown inputs, the equilibrium levels vary in an unpredictable manner and show *behavior* that appears to be independent of the structure. If the gain is >1 or <1, random inputs may mask the resulting growth or decay without destroying the underlying behavior mode. If the gain is +1, random inputs destroy any equilibrium. As the preceding analysis shows, the behavior of the levels is not entirely independent of the structure. The structure preserves a constant of the motion which varies in an unpredictable manner under the influence of unknown inputs. The constant of motion has statistical characteristics more closely related to those of the input than are those of any of the levels in the loop. This can be seen in the Salesman-Backlog model if we add a random input, $u_i$ (i = DRA, BL, S ), to each level equation. The equation for the constant of motion becomes

$$\dot{COM} = (SEM*RS/SS - 1)DRA + (DRAT*u_{DRA} + u_{BL} + SEM*SAT*u_S)$$

Thus, for unity-gain loops, the constant of motion is the integral of a linear combination of the random inputs. If the gain is not unity, the constant of motion is determined by the Delivery Rate Average as well as the random inputs. In this case the auto-correlation of DRA and the cross-correlation of DRA with Salesmen and Backlog will affect the distribution of the constant of motion. The implications of this comment for estimation and control of models containing unity-gain loops remains to be determined. A physical model that is suggestive of some features of unity-gain positive feedback loops is described in Appendix D.

**Genericity**

The generic nature or 'genericity' of a structure is a compound of its ubiquity, simplicity and transparency. A structure is called generic because it represents some elements found in many different systems, it has a relatively small number of levels and its small number of behavior modes depend on only a few combinations of parameters. The dependence of the above analysis on the precise value of the gain around the major loop being unity raises some doubt as to the wide applicability or genericity of the structure. The structure of delays, both material and information delays linked by non-linear auxiliary relations, is quite general. The positive polarity of the major feedback loop is common in systems characterized by a tendency to grow under more or less strict control due to sub-major negative loops. The occurrence of systems with gain of precisely +1 maintained over long periods of time is less probable except for the case of bookkeeping identities. However, we consider unity-gain loops to be a special case of a structure of non-linear delays linked in a positive major loop. This structure can rightly be called generic since it appears in many models of very disparate systems, it has a small set of behavior modes and its behavior is determined by a single, intuitively obvious parameter, the open-loop step gain.

**Structure and Behavior**

Structures of the sort we have described allow us to focus attention on those changes in structure, as embodied in the internal gain factors such as SEM and RS/SS, that change the behavior of the constant of the motion. One should seek to understand the behavior of complex systems by classifying the behavior of simpler, generic systems while making use of the minimum number of concepts. The Salesman-Backlog model shows that it is not always necessary to linearize the model to derive useful insights into the effects of critical parameters on behavior, if one deals with the appropriate generic structure. This paper can be seen as an argument in favor of analyzing systems in terms of generic structures rather than in terms solely of feedback loops.

The dependence of the rate of change of the constant of the motion on the open-loop, step gain allows us to define clearly the *polarity* of a structure in a way that is useful for understanding behavior:

> The polarity of a multi-loop, finite step gain structure contained within a major positive feedback loop (as determined by the product of signs around the loop) is defined by the derivative of the constant of motion with respect to some level in the major loop.

The derivative in question is proportional to (open-loop step gain − 1) so that the polarity is positive if gain>1, negative if gain<1, and neutral if gain=1. Associating the polarity with the *structure*, rather than with a *loop*, is somewhat novel but is clearly justified by the generic nature of the structure. The advantages of this association are to preserve our intuition about the behavior of structures with given polarity. The association works only for the case of *structures* incorporating a positive major loop, i.e. for which the gain>0. When the gain<0, the polarity of the major loop is negative and a simple, intuitive and useful classification of behavior modes for the structure is no longer possible. While the generic structure described above shows overall decay behavior in the case of gain<0, the distinction between pure decay and decaying oscillations is not illuminated by reference to the 'constant' of the motion.

The case of a positive major loop containing one 'bare' integrator (or more) is not included in the class of generic structures since the open-loop step gain is infinite, and there is no constant of the motion. The limiting case of a positive major loop composed entirely of integrators presents some interesting abstract features[2] but is neither realistic nor useful for understanding the behavior of the generic structures considered in this paper. This limiting case and the confusion it has engendered are strong arguments for making realistic generic structures the focus of the analysis of structure and behavior.

**Management Implications**

A knowledge of the constant(s) of motion in a model is of some use in modeling real systems since it reveals what sort of structures are needed to create or destroy the constants of the motion. Such information is useful for managing real systems since it shows:

> (i) the conditions under which sub-models or sub-sectors are rendered independent of other sectors. When the gain is unity, the Salesman-Backlog structure does not exert any stress on other loops. That is, it does not actively determine the goals of other loops in the full model. It reacts to influences arising from other loops by changing its constant of motion instantaneously and adjusting its levels to be in an equilibrium consistent with the current value of the constant of motion.
>
> (ii) the conditions under which a fundamental behavior mode of growth or decay is achieved. When the gain is not unity, the Salesman-Backlog structure has a definite effect on the behavior of the model as can be demonstrated by setting SEM at various values in the full 'Market Growth' model.

In either case, the desired effects can be achieved by attending to only a few critical parameters. Certainly, the critical parameters determining the gain can be found without reference to a constant of the motion. The concept of the 'constant' and its associated structure is useful for decomposing a structure into a core set of major and minor loops whose behavior is understood, and a remainder whose influence on the generic structure can be qualitatively assessed.

The existence of a constant and the attendant equilibrium of the levels means that a stagnating system may have some potential for improvement. A manager can choose his equilibrium within limits imposed by the compensating action of negative feedback loops outside his direct control. In the Salesman-Backlog model, the equilibria shown in Figure 1 are well within the maximum Salesmen and Delivery Rate Average levels imposed by the capacity limit of 12000 un/month. In the full model, the market responds in such a way that the gain of this loop varies around unity. Although the market reaction masks the sensitivity of the unity-gain condition, it does not justify passively accepting the long-term decline in capacity that arises under some management policies. The possibility that a unity-gain structure exist in a larger context should only focus the search for effective control strategies.

Further, one can suggest that having located the unity-gain loops in a system, one should look elsewhere for high-leverage policies to change the behavior. Unity-gain loops do not resist or defeat exogenously imposed inputs so that they are unlikely to be the source of persistent difficulties exemplified by undesirable oscillations or long-term stagnation.

An alternative proposition is that a manager should organise his systems to create unity-gain structures so as to increase his control by increasing the sensitivity of the system to specific inputs. The implementation of this control strategy depends on being able effectively to isolate a sub-system from *uncontrolled* influences. Once isolated, the policy design problem concentrates on making the sub-structure responsive to controlled inputs. The isolation approach is distinct from the robust control approach often favored in system dynamics studies. In the isolation strategy, the robustness is concentrated in the structure that permits isolation. This strategy is not essentially different from the search for high-leverage policies which must be applied with sufficient consistency to achieve their intended effect. The unity-gain structure is another tool to be added to the conventional system design kit.[13]

A high-sensitivity, high-risk control strategy of the isolation type is similar to a proposal by Beer[14] in the context of a discussion of catastrophe theory models. The sensitivity of the constant of motion is due to a singularity of the operator-matrix; the determinant of the (linearized) matrix is zero when the gain is unity. This is a linear phenomenon, unlike the nonlinear singularities of catastrophe theory.

## FURTHER DEVELOPMENT

The treatment of specific cases in this paper somewhat obscures the general structure of the systems under consideration. It is left to a subsequent paper[12] to develop the general case in a more abstract treatment. Some of the issues raised for future study are merely sketched here.

The behavior of the constant of the motion and the fact that its constancy depends on the gain being exactly unity raises the question: What is the behavior of a multi-loop system whose gain may vary around unity at different periods in its evolution. The one-level example of S-shaped growth of a population constrained by limited resources is the simplest such system but the treatment of more complex systems is not complete.

The constant of the motion is found by applying a linear transformation to the operator-matrix. It is an open question under what conditions a constant of the motion can be found for more general structures. Studies of non-linear models of interacting populations show that such a constant exists for a system of negative sub-major loops.[15] The class of systems studied is a generalization of the Lotka-Volterra or Predator-Prey model. A major loop is not necessary for the existence of the constant of the motion. If a major loop does exist, it does so in the form of a positive and a negative major loop which are always simultaneously present. A major loop is rarely encountered in the ecological systems that inspired the generalization since it corresponds to interaction between widely disparate trophic levels. However, the existence of the constant of motion, and the restriction to the physically interesting case of non-negative levels permit proofs of periodicity of two-level systems, of disappearance of species and of statistical properties of populations described by the class in question. Preliminary analysis of the appropriate generalization of the unity-gain structure discussed in this paper indicates that

–Positive feedback loops possess a number of constants of the motion equal to the number of levels in the major loop. These constants appear to be simply re-definitions of the constant of motion in units corresponding to the dimensions of each level.

–Negative feedback loops do not possess constants of the motion in the sense described here so that the concept may be useful primarily for classifying the dynamic nature of multi-loop systems embedded in a major positive loop.

–The existence of the constant of the motion seems to imply that in some sense the system is of lower order (has one state less) than the original system. Useful implications that can be derived from this statement remain to be discovered.

–In some cases the rate of change of the 'constant' of the motion, as constructed by the procedure proposed above, depends on only one or a small number of levels (and hence is not strictly constant). The use of this feature for control or estimation of the model is an open question.

## SUMMARY

By means of several reasonably representative examples of unity-gain positive feedback loops, we have shown that a constant of the motion can be found and that such a constant is a sensitive indicator of the dynamic nature of some multi-loop systems. Some implications of this sensitivity for understanding and controlling such systems are discussed.

The insights from this study, such as they are, are best applied to understanding that positive major loops may show a tendency to seemingly erratic behavior when the gain around the loop approaches unity while the loop is being perturbed in a random or, at least, uncontrolled way. The 'constant of the motion' is a more sensitive indicator of the underlying nature of the loop dynamics than individual levels as the results of the 'Market Growth' model show. This sensitivity allows us to characterize structures containing positive major loops as being of positive, negative or neutral polarity.

## APPENDIX A:

### DYNAMO Equations for the Salesman-Backlog Model

```
L    S.K=S.J+(DT) (SH.JK) SALESMEN
N    S=10
R    SH.KL=(IS.K-S.K)/SAT   SALESMAN HIRING
C    SAT=20   SALESMAN ADJUSTMENT TIME MONTHS
A    IS.K=B.K/2000   INDICATED SALESMEN
A    B.K=DRA.K*12   BUDGET
L    DRA.K=DRA.J+(DT) (DR.JK-DRA.J)/DRAT
                                  DELIVERY RATE AVERAGE
N    DRA=DR
C    DRAT=1   DELIVERY RATE AVERAGING TIME MONTHS
R    DR.KL=12000*PCF.K   DELIVERY RATE UN/MO
A    PCF.K=TABHL(TPCF,DDM.K,0,5,.5)
                                  PRODUCTION CAPACITY FACTOR
T    TPCF=0/.25/.5/.67/.8/.87/.93/.95/.97/.97/1.0
A    DDM.K=BL.K/12000   DELIVERY DELAY MINIMUM
L    BL.K=BL.J+(DT) (OB.JK-DR.JK)   BACKLOG UNITS
N    BL=8000
R    OB.KL=SEM.K*S.K   ORDERS BOOKED UN/MO
A    SEM.K=400+STEP(-233.33,UGT1)+STEP(-66.67,LGT)
X       +STEP(66.67,UGT2)   SALESMAN EFFECTIVENESS
C    UGT1=20,LGT=40,UGT2=60   GAIN STEP TIMES
A    COM.K=DRAT*DRA.K+BL.K+SEM.K*SAT*S.K
                                  CONSTANT OF MOTION
SPEC DT=0.1/LENGTH=90/PLTPER=3
PLOT S=S/BL=B/DRA=D
PLOT COM=C
RUN
```

## APPENDIX B:

### Open-loop Step Gain

For finite open-loop step gain systems, the loop can contain only auxiliaries and delay elements. In particular, no pure integrators can be included since they only accumulate input. As time approaches infinity after application of an input step, the level grows without limit and the gain is infinite. Simple integrators are usually a modelling short-cut which should be replaced by a delay element with a long delay time. The limiting value is derived from the Final Value theorem of Laplace transforms. The output C(s) of a system G(s) with input R(s) is

$$C(s) = R(s)\,G(s)$$

and the limit as time approaches infinity is

$$\lim_{t \to \infty} C(t) = \lim_{s \to 0} s(R(s)\,G(s))$$

For a unit step input

$$R(s) = 1/s.$$

Thus $$\lim_{t \to \infty} C(t) = \lim_{s \to 0} G(s)$$

Note that for a unit impulse input,

$$R(s) = 1$$

and $$C(s) = 1.G(s)$$

In this case, the limit as s–>0 is the same as the step response; however, the limit is not the final value of the impulse response. The final value of the impulse response is rather

$$\lim_{s \to 0} s(1.G(s))$$

which for systems composed of a cascade of delays and auxiliaries is zero.

## APPENDIX C:

### Pricing Model–DYNAMO Equations

```
L    P.K=P.J+(DT)(CP.JK)   PRICE ($/UN)
N    P=100
R    CP.KL=(IP.K-P.K)/TAP   CHANGE IN PRICE ($/UN-WEEK)
A    GM.K=P.K-C   GROSS MARGIN
L    TGM.K=TGM.J+(DT) (GM.J-TGM.J)/TAM
N    TGM=GM   TRADITIONAL GROSS MARGIN ($/UN)
A    IP.K=C+TGM.K   INDICATED PRICE ($/UN)
C    C=80   COST ($/UN)
C    TAP=4   TIME TO ADJUST PRICE WEEKS
C    TAM=2   TIME TO ADJUST MARGIN WEEKS
```

## APPENDIX D

### Interpretation of the Constant of Motion–The Circulating Content

The interpretation of the constant of the motion remains somewhat problematic. In the Job-Vacancy Employment example, the constant was simply the total number of jobs available, whether filled (Employment) or not (Job-Vacancies). In the Pricing model, a strenuous abuse of language would permit us to describe the margins as a rate of spending per unit of product, either real (using the Gross Margin or equivalently the Price) or desired (using the Traditional Gross Margin or equivalently the Indicated Price). Then the constant of motion is the amount of cash per unit 'circulating' around the loop so that each dollar spends TAP time units in the level GM and TAM time units in the level TGM.

The mental model is of a closed tube (representing the major loop) filled with a circulating fluid (the kind of fluid described by the dimensionally similar terms of the constant of motion). The length of the tube is given by the sum of the delay times around the loop and the amount of time a unit of the fluid spends in each level is given by the delay time associated with that level. Thus each term in the constant represents the amount of fluid contributed by the corresponding *level to* the total equilibrium amount of fluid – *the* value of the constant or the *circulating content.* To complete the model, the rate of flow through each level is given by the product of gain factors and the level (excluding the delay time for the level from the gain factors).

For models consisting of linked material delays, the idea of a certain amount of material circulating among the system levels is reasonably cogent. Although this interpretation is somewhat strained in application to systems of mixed information and material delays, it does suggest why unity-gain positive feedback loops are sensitive to exogenous disturbances. The non-viscous and incompressible 'fluid' that forms the circulating content of the loop has no internal damping or shock-absorbing properties so that it reacts instantaneously to externally imposed inputs.

Applying this interpretation to the Salesman-Backlog model, the 'fluid' consists of units of backlog and the flow rate through the level Delivery Rate Average is simply DRA, through the Salesmen is SEM*S and through the Backlog is BL. Analysis of a linearized representation of the Salesman Backlog model shows that the BL term here is the result of dividing by the reciprocal of the slope of PC*PCFo to get the delivery rate and then multiplying by the *same* parameter to get the contribution to the 'circulating content'. The whole Delivery Rate Average term represents the average number of units delivered during a period of DRAT time units. The Salesman term represents the number of units sold by salesmen whose effectiveness is SEM, over a time equal to the adjustment time SAT. The Backlog term represents simply the number of units in the backlog.

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