Technical Note : The Calculation of Average Age of a Group of Items

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Introduction

In some industrial models the firm maintains a number of products at the market and the average age of the product line is one of the factors affecting the firm's sales. Typically we might have:-

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L  PAM.K=PAM.J+DT*(PIR.JK-PWR.JK)          (1)
R  PWR.JK=DELAY3(PIR.JK,PLT)                (2)
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where

PAM = (Products) Number of products in the product line

PIR = (Products/Month) Rate of Introducing new products

PWR = (Products/Month) Rate of Withdrawing old products

PLT = (Months) Average Product Lifetime

We require a method for the calculation of Average Product Age, APA (Months). Very similar problems occur in demographic and manpower planning models.

Approaches to the Problem

Let $\bar{a}_t$ be average product age at time t and let $P_i \ldots P_n$ be the number of products in each of n age classes, with $a_i \ldots a_n$ being the mid points of those age classes. If the a_i are suitably close, e.g. DT apart, then

$$\bar{a}_t = \frac{\sum_1^n P_i a_i}{\sum_1^n P_i} \qquad (3)$$

will be a suitable approximation to the appropriate continuous expression.

One method of calculation would be to use the BOXCAR function, but this has the drawback of replacing the distributed delay of equation (2) by a pipeline delay. An alternative is to use a number of levels, one for the products in each age group, and to model the flows between them, using equation (3) to calculate the resulting average age. Apart from being exceedingly tedious to program and requiring many lines of coding, this has the drawback of requiring some ambitious upper age limit beyond which the products in the 'tail' of the distribution will be assumed to be of the same age. If, for example, PLT = 24, one would need about 30 levels, 30 N equations, and 30 intermediate rates to get even fair numerical accuracy. If the flow from each level to the next is modelled as first order, one has then replaced DELAY3 in equation (2) by a 30th order delay.

It is clear that no really satisfactory method is likely to exist and it would therefore be useful to develop a fairly simple equation which

(i) required only a line or two of coding

(ii) allows one to use in equation (2) whatever delay order is appropriate to the characteristics of the product. (See for example, Coyle (1976), Chapter 2)

(iii) Involves assumptions no more severe than those involved in alternative methods.

A simple equation is given in Coyle (1976) Chapter 9 of

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L  APA.K=APA.J+DT*(1-(PIR.JK*APA.J)/PAM.J)                    (4)
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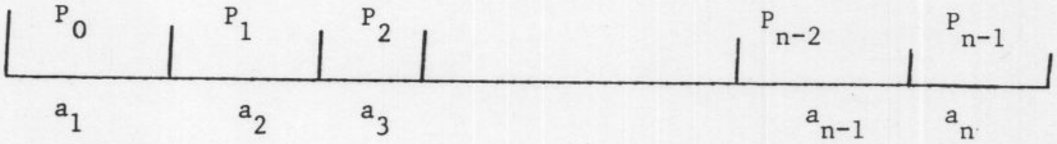
but this assumes that products are introduced at the same rate as they are withdrawn, that the average product is withdrawn at the average product age rather than at the end of the average lifetime, as implied in equation (2), and that products are evenly distributed across the age range. In most cases we are interested in the dynamics generated by the absence of these conditions and we require a more comprehensive equation than (4).

<u>Derivation of the Equation</u>

Let there be n product classes, DT apart in age. Then, at time t, we can draw a diagram

Products	P_1	P_2	P_3		P_{n-1}	P_n
Age	a_1	a_2	a_3		a_{n-1}	a_n

If we approximate the withdrawal process by using the boxcar assumption that a_n = PLT - DT/2 and that all products are withdrawn at that age, then at t + DT we would have



Thus, P new products have come in, P_n have been withdrawn, and all the rest have moved along one category on the age line. Thus

$$\acute{a}_{t+DT} = \frac{\sum_0^{n-1} P_i a_{i+1}}{\sum_0^{n-1} P_i} \qquad (5)$$

However, $a_{i+1} = a_i + DT$ so (5) becomes

$$\bar{a}_{t+DT} = DT + \frac{\sum_0^{n-1} P_i a_i}{\sum_0^{n-1} P_i} \qquad (6)$$

Subtracting (3) from (6) and transposing provides

$$\acute{a}_{t+DT} = \bar{a}_t + DT + \frac{\sum_0^{n-1} P_i a_i}{\sum_0^{n-1} P_i} - \frac{\sum_0^{n-1} P_i a_i}{\sum_1^{n} P_i}$$

Or

$$\acute{a}_{t+DT} = \bar{a}_t + DT + \frac{\sum_0^{n-1} P_i a_i \sum_1^{n} P_i - \sum^{n-1} P_i \sum_1^{n} P_i a_i}{\sum_0^{n-1} P_i \sum_1^{n} P_i} \qquad (7)$$

Consideration of the diagrams for t and t+DT shows that

$$\left.\begin{aligned} \sum_0^{n-1} P_i &= P_0 - P_n + \sum_1^{n} P_i \\ \text{and} \quad \sum_0^{n-1} P_i &= P a_0 - P_n a_n + \sum_1^{n} P_i a_i \end{aligned}\right) \qquad (8)$$

Inserting (8) into (7) and multiplying out leads to

$$\bar{a}_{t+DT} = \bar{a}_t + DT \frac{\sum_1^n P_i (P_0 a_0 - P_n a_n) + \sum_1^n P_i \sum_1^n P_i a_i - (P_0 - P_n) \sum_1^n P_i a_i - \sum_1^n P_i \sum_1^n P_i a_i}{\sum_1^n P_i \left(P_0 - P_n + \sum_1^n P_i\right)}$$

Applying equation (3) and simplifying leads to

$$\bar{a}_{t+DT} = \bar{a}_t + DT + \frac{(P_0 a_0 - P_n a_n) - (P_0 - P_n)\bar{a}_t}{P_0 - P_n + \sum_1^n P_i} \qquad (9)$$

Equation (9) can be translated into DYSMAP by observing that

$\bar{a}_{t+DT}$ = APA.K, $\bar{a}_t$ = APA.J, P_0 = PIR.JK*DT, P_n = PWR.JK*DT,

a_0 = -DT/2, a_n = PLT-DT/2, and $\sum_1^n P_i$ = PAM.J

Thus:-

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L    APA.K = APA.J+DT*(1+(PIR.JK*-DT/2 - PWR.JK*(PLT-DT/2)
X  -   (PIR.JK-PWR.JK)*APA.J)/(PIR.JK*DT-PWR.JK*DT+PAM.J))        (10)
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If the system is even remotely stable, PIR and PWR will be in the same order of majnitude and, further, PAM will be approximately PLT times larger than each of them. We may therefore ignore the terms in DT/2 in the numerator and in DT in the denominator leaving

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L    APA.K = APA.J+DT*(1 - (PWR.JK*PLT+(PIR.JK-PWR.JK)
X    *APA.J)/PAM.J)                                                (11)
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as the result.

In the case where PIR = PWR equation (11) reduces to equation (4) except that (4) had APA in the numerator instead of PLT because (4) assumes that product withdrawals are evenly spread over the age range, whereas (11) concentrates them at the end.

An alternative proof, is that

$$APA = \frac{PY}{PAM}$$

Where PY is the product-years of total age, i.e. $\sum_{1}^{n} P_i a_i$.

Thus

$$\dot{APA} = \frac{\dot{PY}}{PAM} - \frac{PY.\dot{PAM}}{PAM^2}$$

Now $\dot{PY} = PAM - PWR.PLT$

and $\dot{PAM} = (PIR-PWR)$

hence

$$\dot{APA} = 1 - \frac{PWR*PLT + (PIR-PWR).APA}{PAM}$$

which reduces to equation (11)

For a more detailed treatment of the whole question of product introductions, see Coyle and Sharp (1976), problems 8 and 40.

References

Coyle R.G. 'Management System Dynamics', University of Bradford Printed 1976. Forthcoming from John Wiley

Coyle R.G. & Sharp J.A. 'System Dynamics Problems and Cases' System Dynamics Research Group, University of Bradford, 19