Block Diagrams of Generic System Dynamics Models

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Abstract

In system dynamics methodology, a formal mathematical model of a dynamic system consists of a stock-flow diagram and a set of equations. It is possible to simplify and express a system dynamics model as a set of differential equations, which can then be used to obtain the corresponding block diagram for that system dynamics model. In the paper, we obtain simplified differential equations for two system dynamics models and based on the differential equations, we construct two block diagrams. Differential equations serve as a bridge between the two systems modeling perspectives, system dynamics and control theory. In addition, we also show other mathematical forms that can be used to express a dynamic model such as approximate integral equations, difference equations, and integral equations. In Appendix A, a summary of Laplace transforms, transfer functions, and block diagrams are provided as a quick reference. In Appendix B, 18 generic system dynamics models, their simplified differential equations, and their corresponding block diagrams are presented. We carefully formulated SD models and their corresponding block diagrams and verified their behavior by simulating them and by observing the same exact behavior from the SD model and its block diagram. Similar to "differential equations", this paper aims to construct a bridge between control theory and system dynamics.

Keywords: approximate integral equations; block diagram; control theory; differential equations; frequency domain; Laplace transform; stock-flow diagram; system dynamics model.

Introduction

Laplace transform is widely used in control theory, which is a method of converting a set of ordinary differential equations to a set of algebraic equations that can be easily solved. A transfer function is the ratio of a system's output to its input in the Laplace domain, which is also known as the frequency domain (Olivi, 2006). Block diagrams are often used to represent dynamic systems in control theory. Each block in a block diagram has at least two Laplace domain signals connected to it, one input signal and an output signal, and an associated transfer function that transforms the input signal into the output signal. Blocks are connected via their signals (i.e. the output signal generated by a block can be the input to another block). Thus, a complete block diagram represents the dynamic relationship between one input or many inputs to a system and one output or many outputs of that same system (Bequette, 2007; Seborg, 2004).

In system dynamics (SD) methodology, a formal mathematical model of a dynamic system consists of a stock-flow diagram and a set of equations, which together correspond to a set of approximate integral equations. It is also possible to express these models as a set of differential equations (Barlas, 2002; Forrester, 1961 and 1971; IE 533, Unpublished Lecture Notes; IE 550, Unpublished Lecture Notes; Sterman, 2000). As mentioned before, a block diagram represents a set of differential equations in frequency domain. Therefore, it is natural that a block diagram of an SD model can be obtained. Jay Wright Forrester, the founder of SD, developed the field adapting servomechanistic ideas (Forrester, 2007; Lane, 2007). Today, servomechanism theory is known as classical control theory. This paper aims to build a bridge between SD and its roots (i.e. control theory). For this purpose, we constructed block diagrams of well known generic SD models providing details about SD modeling concepts. Such a link between the two fields of dynamic systems will help control theorists to understand SD models and will assist system dynamicists in representing their models using block diagrams, which will hopefully enable them use the analysis methods of control theory. Another aim of this paper is to show different mathematical representations of an SD model. Therefore, after giving the stock-flow diagram and equations of two example models, we also provide their approximate integral equations, difference equations, differential equations, and integral equations.

The first example given in the paper is a basic population model and the second example is a stock management model with three different delay structures; a supply line delay, a decision delay, and a perception delay. In Appendix B, we give SD model, corresponding differential equation(s), and block diagrams of 18 commonly used structures: compounding, draining, first-order linear, production, goal seeking (stock adjustment), capacitated growth, growth with overshoot, a first order and a third order continuous material delay, a first order and a third order continuous information delay, discrete material delay, discrete information delay, oscillating, simple goal setting, epidemic, stock management with a first order and a third supply line delay. Block diagrams that we present are not only exact replicas of their corresponding SD models, but they also include all the details present in the SD models. In addition, we present a summary of Laplace transforms, transfer functions, and block diagrams as a quick reference. We carefully formulated SD models and their corresponding block diagrams and verified their behavior by simulating them and by observing the same exact behavior from the SD model and its block diagram.

A basic population model

Stock-flow diagram of a basic population model is given in Figure 1.

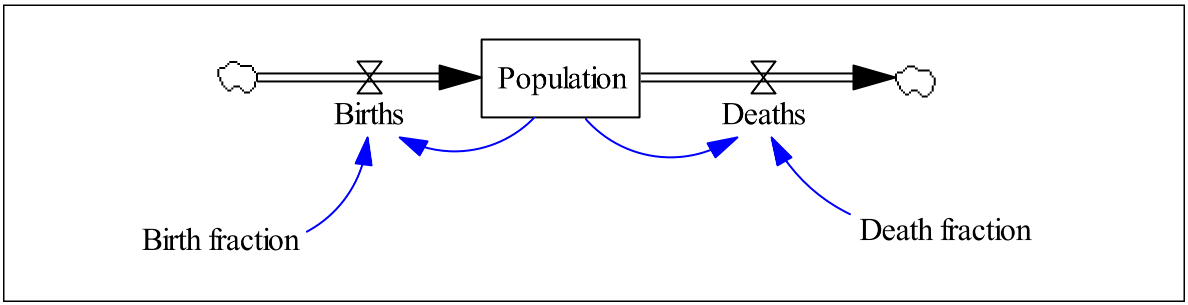

Figure 1. Stock-flow diagram of a basic population model

In the stock-flow diagram given in Figure 1, "Population" is a stock variable, which is an accumulation formed over time. "Population" (p) can only change via "Births" and "Deaths", which are flow variables. There can be one, two, or more than two flows attached to a stock variable. In this simple example, there are only two flows attached to *p*, where "Births" is the inflow and "Deaths" is the outflow. Therefore, "Births" fill in and "Deaths" drain out *p*. "Birth fraction" (*bf*) and "Death fraction" (*df*) are the parameters of the population model, which consists of the stock-flow diagram given in Figure 1 and the equations 1 and 2.

$$Births = bf \times p \tag{1}$$

$$Deaths = df \times p \tag{2}$$

To be able to simulate the model, numerical values must be assigned to *bf*, *df*, and simulation-time-step (*DT*). *bf* and *df* can assume non-negative values and *DT* can assume a value between zero and one. If the value of *DT* is strictly between zero and one, the model corresponds to an approximate integral equation. If the value of *DT* is one, the model corresponds to a difference equation. *DT* cannot be equal to zero.

The approximate integral equation of the basic population model

The relationship between the stock variable, which is *p*, and the flow variables attached to it, which are "Births" and "Deaths", imply Equation 3 (see Figure 1).

$$p_{t+DT} = p_t + (Births - Deaths) \times DT \tag{3}$$

Inserting equations 1 and 2 into Equation 3 and simplifying the equation result in Equation 4, which is the corresponding approximate integral equation of the model (IE 533, Unpublished Lecture Notes).

$$p_{t+DT} = p_t + (bf - df) \times p_t \times DT \tag{4}$$

In continuous time simulation, an approximate integral equation or a set of approximate integral equations are used; the value assigned to *DT* must strictly be less than one and greater than zero (IE 533, Unpublished Lecture Notes; IE 550, Unpublished Lecture Notes).

The difference equation of the basic population model

In discrete time simulation, a difference equation or a set of difference equations are used. Assigning one to *DT* in Equation 4 and simplifying the equation result in Equation 5, which is the corresponding difference equation of the model (IE 533, Unpublished Lecture Notes).

$$p_{t+DT} = (1 + bf - df) \times p_t \tag{5}$$

The differential equation of the basic population model

Equation 4 can be re-written as Equation 6.

$$\frac{p_{t+DT} - p_t}{DT} = (bf - df) \times p \tag{6}$$

Equation 7, which is the corresponding differential equation of the model, is obtained from Equation 6 by taking the limit of *DT* to zero (IE 533, Unpublished Lecture Notes).

$$\frac{dp}{dt} = \lim_{DT \to 0^+} \left(\frac{p_{t+DT} - p_t}{DT} \right) = (bf - df) \times p \tag{7}$$

The integral equation of the basic population model

Equation 7 can be re-written as Equation 8.

$$\int_{p_0}^{p_t} dp = \int_0^t (bf - df) \times p \times dt \qquad (8)$$

Equation 9, which is the corresponding integral equation of the model, is obtained from Equation 8 (IE 533, Unpublished Lecture Notes).

$$p_t = p_0 + \int_0^t (bf - df) \times p \times dt \qquad (9)$$

Block diagram of the basic population model

Block diagram of the basic population model is given in Figure 2. More information on Laplace transforms and block diagrams is provided in Appendix A.

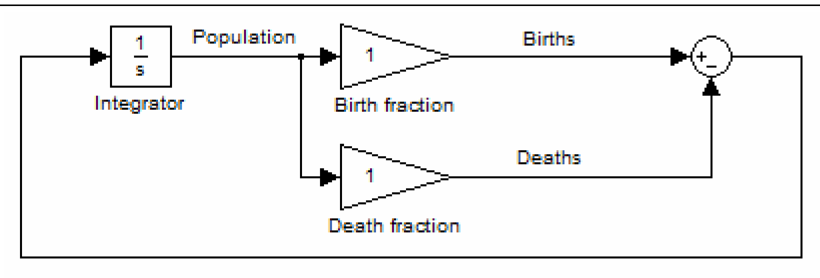

Figure 2. Block diagram of a basic population model

Stock management with three different delay structures: a supply line delay, a decision delay, and perception delay

Stock-flow diagram of a stock management structure with a three different delay structures is given in Figure 3.

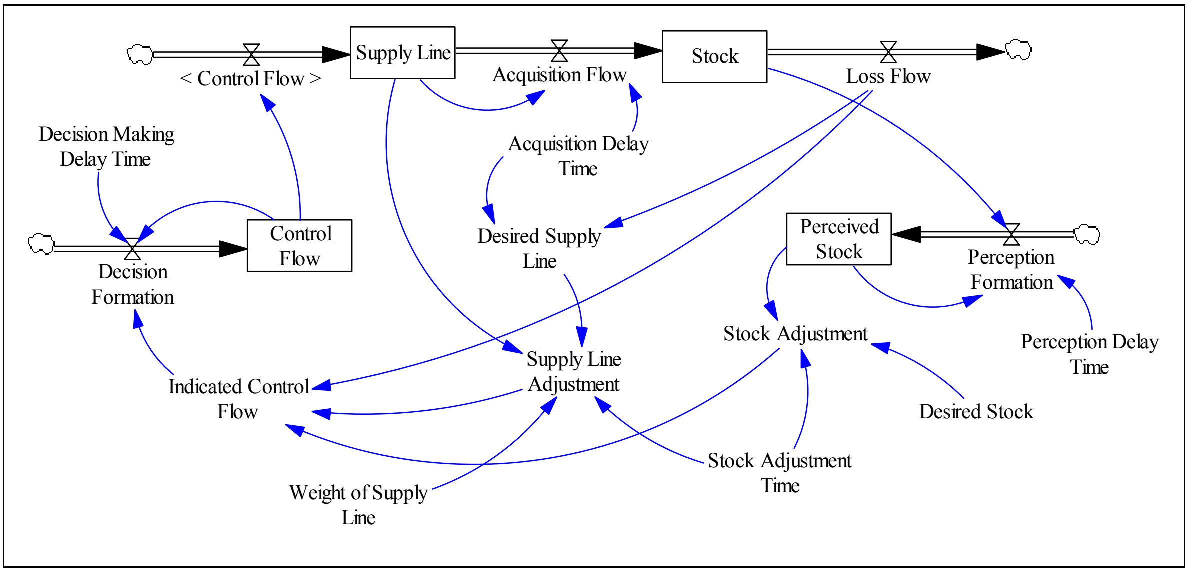

Figure 3. Stock-flow diagram of a stock management structure with three different delay structures

In the stock-flow diagram given in Figure 3, "Supply Line" (*SL*), "Stock" (*S*), "Perceived Stock" (*PS*) and "Control Flow" (*CF*) are stock variables, which are accumulations formed over time. *CF* is, at the same time, a flow variable. The other flow variables are "Acquisition Flow" (*AF*), "Loss Flow" (*LF*), "Perception Formation" (*PF*), and "Decision Formation" (*DF*). *SL* can only change via *CF* and *AF*, *S* can only change via *AF* and *LF*, *PS* can only change via *PF*, *CF* can only change via *DF*. *CF* is the inflow of *SL*, *AF* is the outflow of *SL*, simultaneously, *AF* is the inflow of *S*, *LF* is the outflow of *S*, *PF* is the inflow of *PS*, *DF* is the inflow of *CF*. Therefore, *CF* fill in and *AF* drain out *SL*, *AF* fill in and *LF* drain out *S*, *PF* fill in *PS* and *DF* fill in *CF*. "Indicated Control Flow" (*ICF*), Desired Supply Line (*SL**), "Supply Line Adjustment" (*SLA*), and "Stock Adjustment" (*SA*) are intermediate calculation variables (i.e. auxiliary variables) of the model. "Decision Making Delay Time" *(dmdt)*, "Weight of Supply Line" (*wsl*), "Acquisition Delay Time" (*adt*), "Stock Adjustment Time" (*sat*), "Desired Stock" (*S**) and "Perception Delay Time" (*pdt*) are the parameters of the stock management model that consists of the stock-flow diagram given in Figure 3 and the equations 10-16.

$$ICF = LF + SA + SLA \tag{10}$$

$$SA = \frac{S^* - PS}{sat} \tag{11}$$

$$SLA = wsl \times \frac{SL^* - SL}{sat} \tag{12}$$

$$AF = \frac{SL}{adt} \tag{13}$$

$$SL^* = adt \times LF \tag{14}$$

$$DF = \frac{ICF - CF}{dmdt} \tag{15}$$

$$PF = \frac{S - PS}{pdt} \tag{16}$$

To be able to simulate the model, numerical values must be assigned to *dmdt*, *wsl*, *adt*, *sat*, *S**, *pdt*, and simulation-time-step (*DT*). *dmdt*, *wsl*, *adt*, *sat*, *S**, and *pdt* can assume non-negative values and *DT* can assume a value between zero and one. If the value of *DT* is strictly between zero and one, the model corresponds to a set of approximate integral equations. If the value of *DT* is one, the model corresponds to a set of difference equations. *DT* cannot be equal to zero.

The set of approximate integral equations of the stock management model with three different delay structures

The relationship between the stock variable, which are *S*, *SL*, *CF*, *PS*, and the flow variables attached to it, which are *AF*, *LF*, *CF*, *DF*, and *PF*, imply equations 17-20 (see Figure 3).

$$S_{t+DT} = S_t + (AF - LF) \times DT \tag{17}$$

$$SL_{t+DT} = SL_t + (CF - AF) \times DT \tag{18}$$

$$CF_{t+DT} = CF_t + (DF) \times DT \tag{19}$$

$$PS_{t+DT} = PS_t + (PF) \times DT \tag{20}$$

Inserting equations 10-16 into equations 17-20 and simplifying the equations result in equations 21-24, which are the corresponding set of approximate integral equations of the model (IE 533, Unpublished Lecture Notes).

$$S_{t+DT} = S_t + \left(\frac{SL}{adt} - LF \right) \times DT \tag{21}$$

$$SL_{t+DT} = SL_t + \left(CF - \frac{SL}{adt} \right) \times DT \tag{22}$$

$$CF_{t+DT} = CF_t + \left(\frac{LF + \frac{S^* - PS}{sat} + wsl \times \frac{(adt \times LF - SL)}{sat} - CF}{dmdt} \right) \times DT \tag{23}$$

$$PS_{t+DT} = PS_t + \left(\frac{S - PS}{pdt} \right) \times DT \tag{24}$$

The set of difference equations of the stock management model with three different delay structures

Assigning one to DT in equations 21-24 and simplifying the equations result in equations 25-28, which are the corresponding set of difference equations of the model (IE 533, Unpublished Lecture Notes).

$$S_{t+DT} = S_t + \left(\frac{SL}{adt} - LF \right) \tag{25}$$

$$SL_{t+DT} = SL_t + \left(CF - \frac{SL}{adt} \right) \tag{26}$$

$$CF_{t+DT} = CF_t + \left(\frac{LF + \frac{S^* - PS}{sat} + wsl \times \frac{(adt \times LF - SL)}{sat} - CF}{dmdt} \right) \tag{27}$$

$$PS_{t+DT} = PS_t + \left(\frac{S - PS}{pdt} \right) \tag{28}$$

The set of differential equations of the stock management model with three different delay structures

Equations 21-24 can be re-written as equations 29-32.

$$\frac{S_{t+DT} - S_t}{DT} = \left(\frac{SL}{adt} - LF \right) \tag{29}$$

$$\frac{SL_{t+DT} - SL_t}{DT} = \left(CF - \frac{SL}{adt} \right) \tag{30}$$

$$\frac{CF_{t+DT} - CF_t}{DT} = \left(\frac{LF + \frac{S^* - PS}{sat} + wsl \times \frac{(adt \times LF - SL)}{sat} - CF}{dmdt} \right) \tag{31}$$

$$\frac{PS_{t+DT} - PS_t}{DT} = \left(\frac{S - PS}{pdt} \right) \tag{32}$$

Equations 33-36, which are the corresponding set of differential equations of the model, are obtained from equations 29-32 by taking the limit of *DT* to zero (IE 533, Unpublished Lecture Notes).

$$\frac{dS}{dt} = \lim_{DT \to 0^+} \left(\frac{S_{t+DT} - S_t}{DT} \right) = \left(\frac{SL}{adt} - LF \right) \tag{33}$$

$$\frac{dSL}{dt} = \lim_{DT \to 0^+} \left(\frac{SL_{t+DT} - SL_t}{DT} \right) = \left(CF - \frac{SL}{adt} \right) \tag{34}$$

$$\frac{dCF}{dt} = \lim_{DT \to 0^+} \left(\frac{CF_{t+DT} - CF_t}{DT} \right) = \left(\frac{LF + \frac{S^* - PS}{sat} + wsl \times \frac{(adt \times LF - SL)}{sat} - CF}{dmdt} \right) \tag{35}$$

$$\frac{dPS}{dt} = \lim_{DT \to 0^+} \left(\frac{PS_{t+DT} - PS_t}{DT} \right) = \left(\frac{S - PS}{pdt} \right) \tag{36}$$

The set of integral equations of the stock management model with three different delay structures

Equations 33-36 can be re-written as equations 37-40

$$\int_{S_0}^{S_t} dS = \int_0^t \left(\frac{SL}{adt} - LF \right) \times dt \tag{37}$$

$$\int_{SL_0}^{SL_t} dSL = \int_0^t \left(CF - \frac{SL}{adt} \right) \times dt \tag{38}$$

$$\int_{CF_0}^{CF_t} dCF = \int_0^t \left(\frac{LF + \frac{S^* - PS}{sat} + wsl \times \frac{(adt \times LF - SL)}{sat} - CF}{dmdt} \right) \times dt \tag{39}$$

$$\int_{PS_0}^{PS_t} dPS = \int_0^t \left(\frac{S - PS}{pdt} \right) \times dt \tag{40}$$

Equations 41-44, which are the corresponding set of integral equations of the model, are obtained from equations 37-40 (IE 533, Unpublished Lecture Notes).

$$S_t = S_0 + \int_0^t \left(\frac{SL}{adt} - LF \right) \times dt \tag{41}$$

$$SL_t = SL_0 + \int_0^t \left(CF - \frac{SL}{adt} \right) \times dt \tag{42}$$

$$CF_t = CF_0 + \int_0^t \left(\frac{LF + \frac{S^* - PS}{sat} + wsl \times \frac{(adt \times LF - SL)}{sat} - CF}{dmdt} \right) \times dt \tag{43}$$

$$PS_t = PS_0 + \int_0^t \left(\frac{S - PS}{pdt} \right) \times dt \tag{44}$$

Block diagram of the stock management model with three different delay structures

Block diagram of the stock management structure with third order delay is given in Figure 4. More information on Laplace transforms and block diagrams is provided in Appendix A.

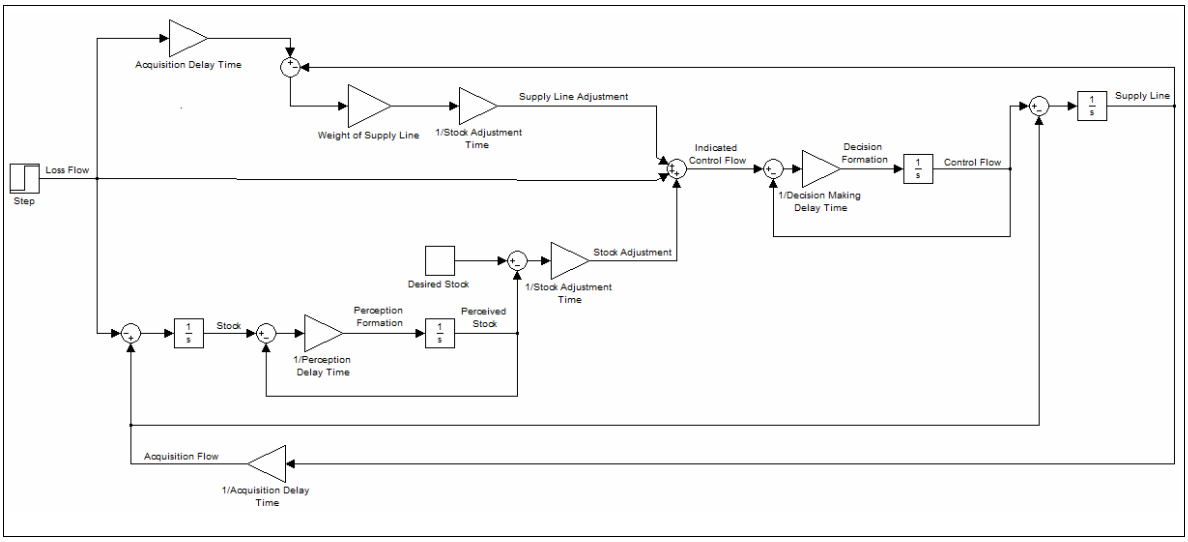

Figure 4. Block diagram of the stock management structure with third order delay

Conclusion

In this paper, block diagrams of well known generic SD models are constructed. Such a link between system dynamics and control theory will help control theorists to understand SD models and will assist system dynamicists in representing their models using block diagrams. This paper presents the preliminary work of an ongoing master thesis, which mainly focuses on modeling and analyzing inventory control systems. The plan is to use both system dynamics and control theory as methodological approaches.

In the paper including its Appendix B, we present twenty different SD models and their corresponding block diagrams. Block diagrams that we present are not only the exact replicas of their corresponding SD models, but they also include all the details present in the SD models. We carefully formulated SD models and their corresponding block diagrams. We simulated the SD models using Vensim and their corresponding block diagrams by using Matlab's Simulink and observe the same exact behavior from each one of the SD models and their corresponding block diagrams.

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Appendix A: Laplace transforms, transfer functions, and block diagrams

In this appendix, we present a summary of Laplace transforms, transfer functions, and block diagrams.

Laplace transform method

Laplace transform of a time domain function, *f(t)*, is given in Equation A.1 (Bequette, 2007).

$$F(s) = L[f(t)] = \int_0^\infty f(t) \times e^{-st} \times dt \tag{A.1}$$

where s is a complex variable.

An example of Laplace transformation: exponential function

In this section, the Laplace transform of an exponential function is obtained as an example (Equation A.6).

$$L[e^{-at}] = \int_0^\infty e^{-at} \times e^{-st} \times dt \tag{A.2}$$

where a is a constant. Equation A.2 can be re-written as Equation A.3.

$$L[e^{-at}] = \lim_{b \to \infty} \int_0^b e^{-(s+a) \times t} \times dt \tag{A.3}$$

$$L[e^{-at}] = \lim_{b \to \infty} \left[\frac{e^{-(s+a) \times t}}{-(s+a)} \right]_0^b \tag{A.4}$$

$$L\left[e^{-at}\right] = \lim_{b\to\infty}\left[\frac{e^{-(s+a)\times b}}{-(s+a)} + \frac{1}{s+a}\right] \tag{A.5}$$

Finally, Equation A.6, which is the Laplace transform of an exponential function, is obtained from Equation A.5.

$$L\left[e^{-at}\right] = \frac{1}{s+a} \tag{A.6}$$

The Generic form of the Laplace transform of a time delayed function (pure delay)

If $f(t)$ represents the value of a particular function at time t, its time delayed version, $f(t-\theta)$, represents the value of that function at time $t-\theta$, where θ, which is a positive constant, is the duration of the delay. In this section, the generic Laplace transform of a delayed function is derived (Equation A.12).

$$L\left[f(t-\theta)\right] = \int_0^\infty f(t-\theta)\times e^{-st}\times dt \tag{A.7}$$

Equation A.7 can be re-written as Equation A.8.

$$L\left[f(t-\theta)\right] = \int_0^\infty f(t-\theta)\times e^{-s(t-\theta+\theta)}\times dt \tag{A.8}$$

$$L\left[f(t-\theta)\right] = \int_0^\infty f(t-\theta)\times e^{-s(t-\theta)}\times e^{-s\theta}\times dt \tag{A.9}$$

$$L\left[f(t-\theta)\right] = e^{-s\theta}\times\int_0^\infty f(t-\theta)\times e^{-s(t-\theta)}\times d(t-\theta) \tag{A.10}$$

If a change of variables, $t^* = t - \theta$ is used to integrate the function (Equation A.10), Equation A.11 is obtained.

$$L[f(t-\theta)] = e^{-s\theta} \times \int_0^{\infty} f(t^*) \times e^{-st^*} \times dt^* \tag{A.11}$$

Finally, Equation A.12, which is the generic form of the Laplace transform of a time-delayed function, is obtained from Equation A.11.

$$L[f(t-\theta)] = e^{-s\theta} \times F(s) \tag{A.12}$$

According to Equation A.12, Laplace transform of the delayed version of a function equals to $e^{-s\theta}$ times the Laplace transform of that function (Bequette, 2007; Seborg et al., 2004).

The generic form of the Laplace transform of a first order derivative

The generic form of the Laplace transform of a first order derivative of a function is obtained by using integration by parts technique (Equation A.17).

$$L\left[\frac{df(t)}{dt}\right] = \int_0^{\infty} \frac{df(t)}{dt} \times e^{-st} \times dt \tag{A.13}$$

Equation A.13 can be re-written as Equation A.14.

$$L\left[\frac{df(t)}{dt}\right] = \lim_{b \to \infty} \int_0^{b} e^{-st} \times \frac{df(t)}{dt} \times dt \tag{A.14}$$

$$L\left[\frac{df(t)}{dt}\right] = \lim_{b \to \infty} \left[f(t) \times e^{-st} \Big|_0^b + s \times \int_0^{b} f(t) \times e^{-st} \times dt \right] \tag{A.15}$$

$$L\left[\frac{df(t)}{dt}\right] = \lim_{b\to\infty}\left[f(b)\times e^{-sb} - f(0) + s\times\int_0^b f(t)\times e^{-st}\times dt\right] \quad \text{(A.16)}$$

Finally, Equation A.17, which is the generic form of the Laplace transform of a first order derivative of a function, is obtained from Equation A.16.

$$L\left[\frac{df(t)}{dt}\right] = s\times L[f(t)] - f(0) \quad \text{(A.17)}$$

Laplace transforms of commonly used time domain functions

Laplace transforms are used for solving most dynamic problems and, in solving such a problem, Laplace transform tables are usually used to save time. Accordingly, a Laplace transform table for some of the common functions is also provided in this Appendix (Table A.1). For a more comprehensive Laplace transform table, see, for example, Bequette (2007) or Seborg et al. (2004).

Table A.1. Laplace Transforms of Common Time Domain Functions

Time domain function	**Laplace domain function**
$f(t)$	$F(s)$
$\delta(t)$ (Equation A.18)	1
$S(t)$ (Equation A.20)	$\frac{1}{s}$
a	$\frac{a}{s}$
$f(t-\theta)$	$e^{-\theta s} \times F(s)$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{-at}	$\frac{1}{s+a}$
$t \times e^{-at}$	$\frac{1}{(s+a)^2}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$\frac{df(t)}{dt}$	$s \times L[f(t)] - f(0)$
$\frac{d^n f(t)}{dt}$	$s^n \times F(s) - s^{n-1} \times f(0) - s^{n-2} \times f'(0) - s^{n-3} \times f''(0) - \cdots - f^{(n-1)}(0)$

Unit impulse ($\delta(t)$) is defined by Equation A.18 and its integral from negative infinity to positive infinity, which is equal to one, is given in Equation A.19.

$$\delta(t) = \lim_{\varepsilon \to 0^+} \begin{Bmatrix} \frac{1}{\varepsilon} & \text{for} & 0 \le t \le \varepsilon \\ 0 & \text{for} & t > \varepsilon \end{Bmatrix} \quad \text{(A.18)}$$

$$\int_{-\infty}^{+\infty} \delta(t) \times dt = \int_{0}^{\varepsilon} \delta(t) \times dt = \lim_{\varepsilon \to 0^+} \int_{0}^{\varepsilon} \frac{1}{\varepsilon} \times dt = 1 \quad \text{(A.19)}$$

Unit step ($S(t)$) is defined by Equation A.20.

$$S(t) = \begin{Bmatrix} 0 & \text{for} & t < 0 \\ 1 & \text{for} & t \geq 0 \end{Bmatrix} \quad \text{(A.20)}$$

It is also possible to use a Laplace transform table (Table A.1) to obtain the inverse Laplace transform of a Laplace domain function, which is defined by Equation A.21.

$$L^{-1}[F(s)] = f(t) \quad \text{(A.21)}$$

Note that the inverse Laplace transform of the Laplace transform of a time domain function is itself (Equation A.22).

$$f(t) = L^{-1}[L[f(t)]] \quad \text{(A.22)}$$

Solving linear differential equations using Laplace transforms: an example first order equation

To solve a differential equation with Laplace transform, Laplace transform of both sides of the differential equation must be taken. Then, the resulting algebraic equation must be solved for $L[f(t)]$. Finally, the inverse transform must be taken by using Laplace transform table.

An example first order differential equation is given below (Equation A.23):

$$\frac{dx(t)}{dt} - x(t) = e^t \quad \text{(A.23)}$$

Initial condition is $x(0) = a$, where a is a constant value.

Taking the Laplace of Equation A.23 gives Equation A.24.

$$L\left[\frac{dx(t)}{dt} - x(t)\right] = L\left[e^t\right] \tag{A.24}$$

$$L\left[\frac{dx(t)}{dt}\right] - L\left[x(t)\right] = L\left[e^t\right] \tag{A.25}$$

Equation A.26 is obtained from Equation A.25 using Table A.1.

$$\left(s \times L\left[x(t)\right] - x(0)\right) - L\left[x(t)\right] = \frac{1}{s-1} \tag{A.26}$$

Equation A.26 can be re-written as Equation A.27.

$$\left(s-1\right) \times L\left[x(t)\right] - a = \frac{1}{s-1} \tag{A.27}$$

Solving for $L[x(t)]$ gives Equation A.28.

$$L\left[x(t)\right] = \frac{1}{\left(s-1\right)^2} + \frac{a}{\left(s-1\right)} \tag{A.28}$$

Equation A.29 is obtained by inverting Equation A.28 to the time domain using Laplace transform table (Table A.1).

$$x(t) = t \times e^t + a \times e^t \tag{A.29}$$

Solving linear differential equations using Laplace transforms: an example set of first order equations

An example of a set of first order differential equations is given below (equations A.30 and A.31):

$$\frac{dx_1(t)}{dt} + x_2(t) = 1 \tag{A.30}$$

$$x_1(t) - \frac{dx_2(t)}{dt} = 0 \tag{A.31}$$

Initial conditions are $x_1(0) = a_1$ and $x_2(0) = a_2$, where a_1 and a_2 are constants.

Taking the Laplace of equations A.30 and A.31 gives equations A.32 and A.33.

$$L\left[\frac{dx_1(t)}{dt}\right] + L[x_2(t)] = L[1] \tag{A.32}$$

$$L[x_1(t)] - L\left[\frac{dx_2(t)}{dt}\right] = L[0] \tag{A.33}$$

Equations A.34 and A.35 are obtained from equations A.32 and A.33 using Table A.1.

$$s \times X_1(s) - x_1(0) + X_2(s) = \frac{1}{s} \tag{A.34}$$

$$X_1(s) - s \times X_2(s) + x_2(0) = 0 \tag{A.35}$$

Equations A.34 and A.35 can be re-written as equations A.36 and A.37.

$$s \times X_1(s) + X_2(s) = \frac{1}{s} + a_1 \tag{A.36}$$

$$X_1(s) - s \times X_2(s) = -a_2 \tag{A.37}$$

Equation A.38 is obtained from Equation A.37.

$$X_1(s) = -a_2 + s \times X_2(s) \tag{A.38}$$

Equation A.39 is obtained by inserting Equation A.38 into Equation A.36.

$$-a_2 \times s + s^2 \times X_2(s) + X_2(s) = \frac{1}{s} + a_1 \tag{A.39}$$

Solving for $X_2(s)$ gives Equation A.40.

$$X_2(s) = \frac{1}{s \times \left(1 + s^2\right)} + \frac{a_1}{s^2 + 1} + \frac{a_2 \times s}{s^2 + 1} \tag{A.40}$$

Equation A.41 is obtained from Equation A.40.

$$X_2(s) = \frac{1}{s} - \frac{\left(1 - a_2\right) \times s}{s^2 + 1} + \frac{a_1}{s^2 + 1} \tag{A.41}$$

Equation A.42 is obtained by inserting Equation A.41 into Equation A.38.

$$X_1(s) = -a_2 + s \times \left[\frac{1}{s} - \frac{\left(1 - a_2\right) \times s}{s^2 + 1} + \frac{a_1}{s^2 + 1}\right] \tag{A.42}$$

Equation A.42 is simplified to Equation A.43.

$$X_1(s) = -a_2 + 1 - \left(1 - a_2\right) \times \left[1 - \frac{1}{s^2 + 1}\right] + \frac{a_1 \times s}{s^2 + 1} \tag{A.43}$$

$$X_1(s) = \frac{(1-a_2)}{s^2+1} + \frac{a_1 \times s}{s^2+1} \qquad \text{(A.44)}$$

Using Table A.1, the two Laplace domain functions (equations A.41 and A.44) is inverted to the time domain, which are given below:

$$x_1(t) = (1-a_2) \times \sin(t) + a_1 \times \cos(t) \qquad \text{(A.45)}$$

$$x_2(t) = 1-(1-a_2) \times \cos(t) + a_1 \times \sin(t) \qquad \text{(A.46)}$$

Solving linear differential equations using Laplace transforms: an example second order equation

An example second order linear differential equation is given below (Equation A.47):

$$\frac{d^2x(t)}{dt^2} - 4 \times \frac{dx(t)}{dt} + 4 \times x(t) = e^{-2\times t} \qquad \text{(A.47)}$$

Initial conditions are $x(0) = a_1$ and $\dot{x}(0) = a_2$, where a_1 and a_2 are constants.

Taking the Laplace of Equation A.47 gives Equation A.48.

$$L\left[\frac{d^2x(t)}{dt^2}\right] - 4 \times L\left[\frac{dx(t)}{dt}\right] + 4 \times L[x(t)] = L\left[e^{-2\times t}\right] \qquad \text{(A.48)}$$

Equation A.49 is obtained from Equation A.48 using Table A.1.

$$\left(s^2 \times X(s) - s \times x(0) - \dot{x}(0)\right) - 4 \times \left(s \times X(s) - x(0)\right) + 4 \times X(s) = \frac{1}{s+2} \qquad \text{(A.49)}$$

Equation A.49 can be re-written as Equation A.50.

$$\left(s^2 - 4\times s + 4\right)\times X(s) - a_1 \times s - a_2 + 4\times a_1 = \frac{1}{s+2} \qquad \text{(A.50)}$$

Solving for $X(s)$ gives Equation A.51.

$$X(s) = \frac{1}{\left(s^2 - 4\times s + 4\right)\times\left(s+2\right)} + \frac{a_1 \times s}{s^2 - 4\times s + 4} + \frac{\left(a_2 - 4\times a_1\right)}{s^2 - 4\times s + 4} \qquad \text{(A.51)}$$

Equation A.52 is obtained from Equation A.51.

$$X(s) = \frac{1}{16\times\left(s+2\right)} + \left(a_1 - \frac{1}{16}\right)\times\frac{1}{s-2} + \left(\frac{1}{4} - 2\times a_1 + a_2\right)\times\frac{1}{\left(s-2\right)^2} \qquad \text{(A.52)}$$

Equation A.53 is obtained by inverting Equation A.52 to the time domain using Laplace transform table (Table A.1).

$$x(t) = \frac{1}{16}\times e^{-2\times t} + \left(a_1 - \frac{1}{16}\right)\times e^{2\times t} + \left(\frac{1}{4} - 2\times a_1 + a_2\right)\times t\times e^{2\times t} \qquad \text{(A.53)}$$

Transfer functions and block diagrams

Transfer function of a dynamic system is the ratio of the output variable to the input variable in the Laplace domain. In general, $g(s)$ is used to represent a transfer function that is defined in Equation A.54 (Bequette, 2007; Seborg et al., 2004).

$$g(s) = \frac{y(s)}{u(s)} \qquad \text{(A.54)}$$

where $u(s)$ is the input variable and y(s) is the output variable in the Laplace domain.

Block diagrams

Transfer functions are often used in block diagrams. The relationship between input and output defined by Equation A.54 is depicted in Figure A.1.

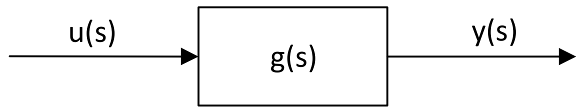

Figure A.1 The most basic block diagram representing an input-output relationship in the Laplace domain

Block diagrams have three main types of elements which are signals, unary operator blocks, and m-ary (i.e., many-ary) operator blocks given in Figure A.2.

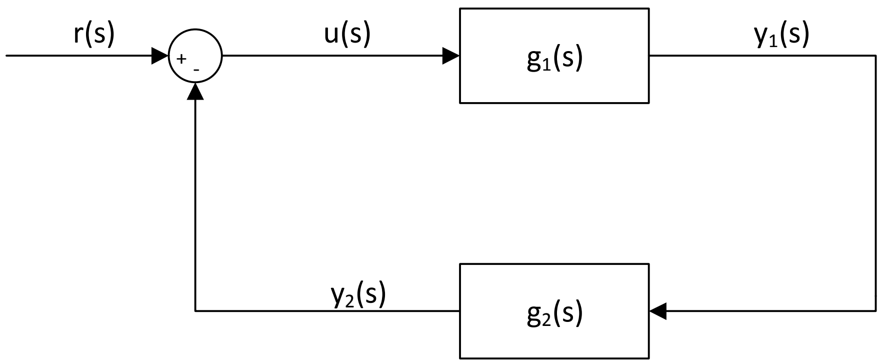

Figure A.2 A block diagram with two blocks

In the block diagram representation given in Figure A.2, $r(s)$, $u(s)$, $y_1(s)$, and $y_2(s)$ are signals. $u(s)$ is the input signal and $y_1(s)$ is the output signal of $g_1(s)$. $y_1(s)$ is the input signal and $y_2(s)$ is the output signal of $g_2(s)$. $g_1(s)$ and $g_2(s)$ are unary operator blocks which operates on the input signals connected to them with the transfer functions to form the output signals. $r(s)$ and $y_2(s)$ are the input variables and $u(s)$ is the output variable of the summation block which is an m-ary operator block. An m-ary operator block is shown as a circle and has two or more input variables and a single output variable (Wescott, 2006).

Reduced block diagrams

A generic example is given in Figure A.3.

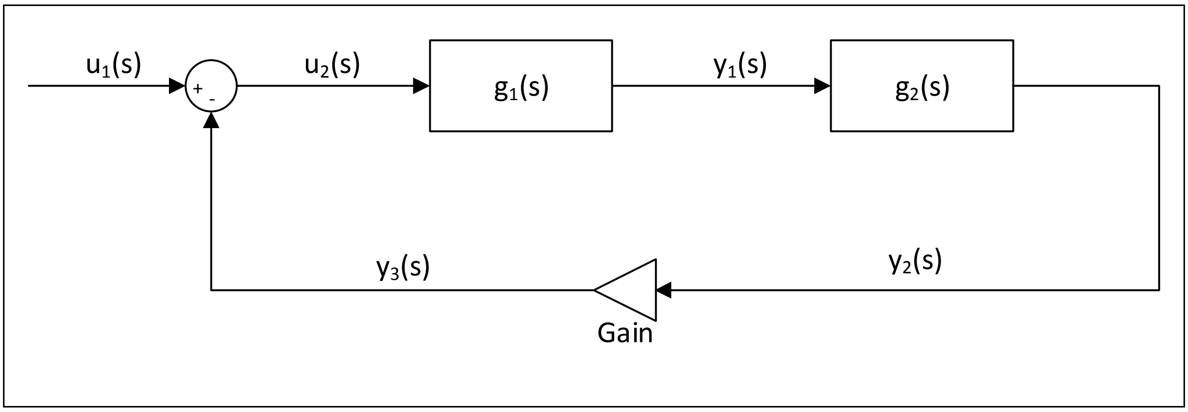

Figure A.3 A generic example of block diagram

The relationship between inputs and outputs of the system are given in equations A.55 and A.56, A.57 and A.58.

$$g_1(s) = \frac{y_1(s)}{u_2(s)} \tag{A.55}$$

$$g_2(s) = \frac{y_2(s)}{y_1(s)} \tag{A.56}$$

$$y_3(s) = y_2(s) \times Gain \tag{A.57}$$

$$u_1(s) - y_s(s) = u_2(s) \tag{A.58}$$

Combining $g_1(s)$ and $g_2(s)$ into a single transfer function, $g_3(s)$ is obtained which is given in Equation A.59.

$$g_3(s) = g_1(s) \times g_2(s) = \frac{y_1(s)}{u_2(s)} \times \frac{y_2(s)}{y_1(s)} = \frac{y_2(s)}{u_2(s)} \tag{A.59}$$

Equation A.58 can be rewritten as Equation A.60.

$$u_1(s) - y_2(s) \times Gain = \frac{y_2(s)}{g_1(s) \times g_2(s)} \tag{A.60}$$

$$u_1(s) = \frac{y_2(s)}{g_1(s) \times g_2(s)} + y_2(s) \times Gain \tag{A.61}$$

$$u_1(s) = y_2(s) \times \left[\frac{1 + Gain \times g_1(s) \times g_2(s)}{g_1(s) \times g_2(s)} \right] \tag{A.62}$$

Overall transfer function of the process, $g_4(s)$ is given in Equation A.63.

$$g_4(s) = \frac{y_2(s)}{u_1(s)} = \frac{g_1(s) \times g_2(s)}{1 + Gain \times g_1(s) \times g_2(s)} \tag{A.63}$$

The reduced form of the block diagram given in Figure A.3 is depicted in Figure A.4, which is also a block diagram and an equivalent of the diagram given in Figure A.3.

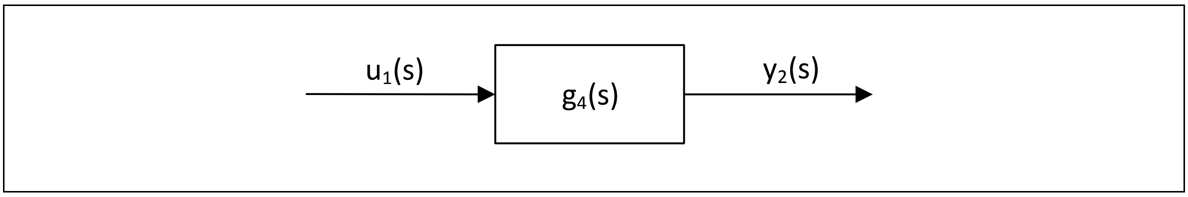

Figure A.4 Reduced form of the example block diagram

Appendix B: The corresponding block diagrams of basic system dynamics models

We first convert generic system dynamics model structures to differential equations and, later, we obtain corresponding block diagrams of these structures from the differential equations. To save space, the derivation process is not provided, but only the resulting differential equations and block diagrams are given. If the reader is interested in the derivation process, she can read the paper and Appendix A and carry out derivations herself. Note that a comprehensive model usually contains one or many of these generic structures. Moreover, a constant of a simpler structure may turn into a variable, even into a state variable, in a more complex model.

Compounding structure

Stock-flow diagram of a compounding structure is given in Figure B.1.

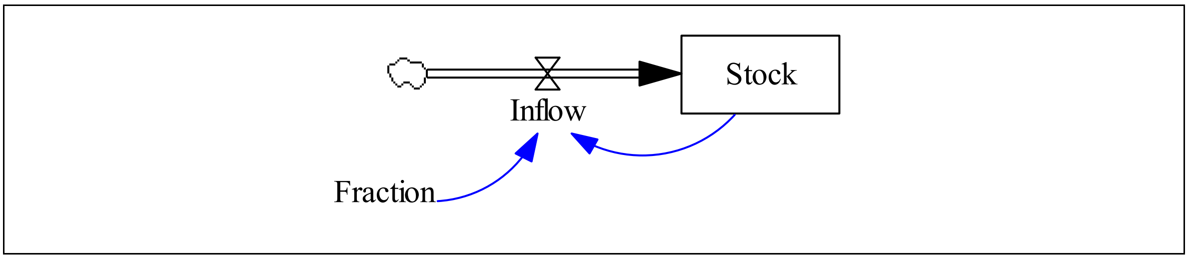

Figure B.1 Stock-flow diagram of the compounding structure

The inflow equation of the model is given in Equation B.1.

$$Inflow = Fraction \times Stock \tag{B.1}$$

where “Fraction” is a nonnegative constant value.

The diagram in Figure B.1 and Equation B.1 define a compounding structure. The simplified differential equation that corresponds to this structure is given in Equation B.2.

$$\frac{dStock}{dt} = Inflow = Fraction \times Stock \qquad (B.2)$$

Block diagram of the compounding structure is given in Figure B.2.

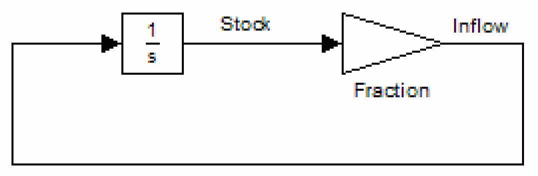

Figure B.2 Block diagram of the compounding structure

Draining structure

Stock-flow diagram of a draining structure is given in Figure B.3.

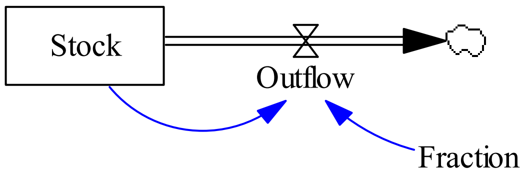

Figure B.3 Stock-flow diagram of the draining structure

The outflow equation of the model is given in Equation B.3.

$$Outflow = Fraction \times Stock \qquad (B.3)$$

where "Fraction" is a nonnegative constant value.

The diagram in Figure B.3 and Equation B.3 define a draining structure. The simplified differential equation that corresponds to this structure is given in Equation B.4.

$$\frac{dStock}{dt} = -Outflow = -Fraction \times Stock \qquad (B.4)$$

Block diagram of the draining structure is given in Figure B.4.

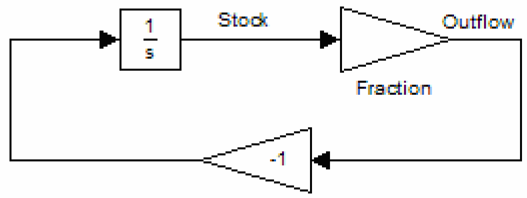

Figure B.4 Block diagram of the draining structure

First order linear structure

Stock-flow diagram of a first order linear structure is given in Figure B.5.

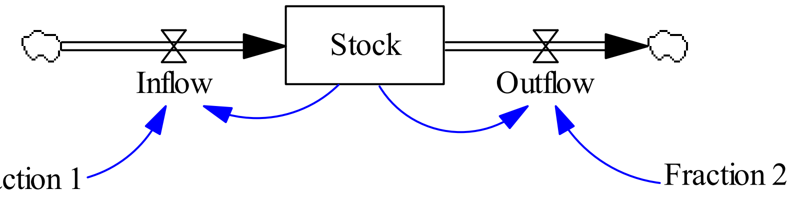

Figure B.5 Stock-flow diagram of the first order linear structure

The inflow and outflow equations of the model are given in equations B.5 and B.6.

$$Inflow = Fraction\ 1 \times Stock \qquad (B.5)$$

$$Outflow = Fraction\ 2 \times Stock \qquad (B.6)$$

where “Fraction 1” and “Fraction 2” are nonnegative constant values.

The diagram in Figure B.5 and equations B.5 and B.6 define a first order linear structure. The simplified differential equation that corresponds to this structure is given in Equation B.7.

$$\frac{dStock}{dt} = Inflow - Outflow = (Fraction\ 1 - Fraction\ 2) \times Stock \qquad (B.7)$$

Block diagram of the first order linear structure is given in Figure B.6.

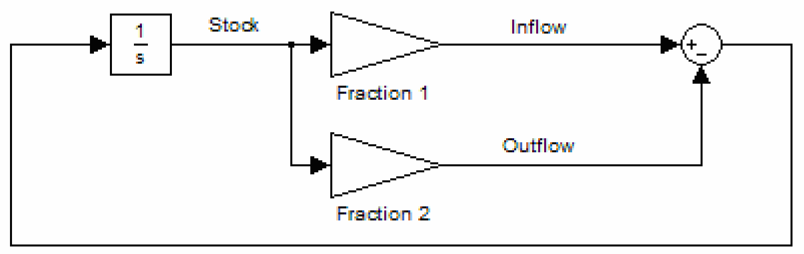

Figure B.6 Block diagram of the first order linear structure

Production structure

Stock-flow diagram of a production structure is given in Figure B.7.

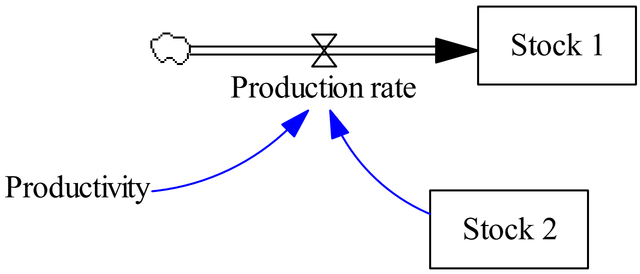

Figure B.7 Stock-flow diagram of the production structure

The inflow equation of the model is given in Equation B.8.

$$Production\ \ rate = Productivity \times Stock\ 2 \qquad \text{(B.8)}$$

where “Productivity” is a nonnegative constant value.

The diagram in Figure B.7 and Equation B.8 define a production structure. The simplified differential equation that corresponds to this structure is given in Equation B.9.

$$\frac{dStock\ 1}{dt} = Production\ \ rate = Productivity \times Stock\ 2 \qquad \text{(B.9)}$$

Flows are not connected to the state variable named “Stock 2” because the focus of this structure is on representing “Production rate” flow. Similar to other simple model structures, the production structure usually is a part of a more comprehensive model.

Block diagram of the production structure is given in Figure B.8.

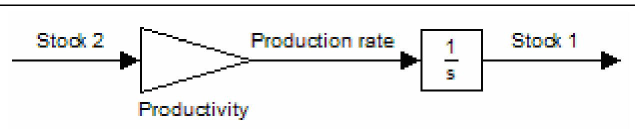

Figure B.8 Block diagram of the production structure

Goal seeking structure

Stock-flow diagram of a goal seeking structure, which is also known as stock adjustment structure, is given in Figure B.9.

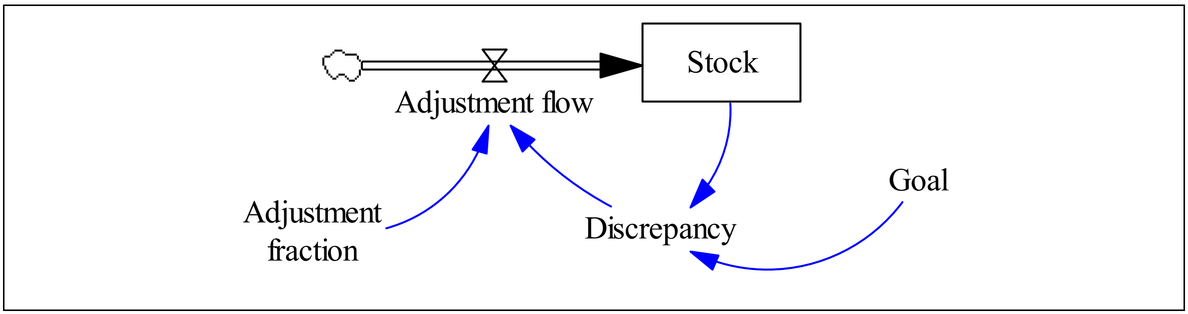

Figure B.9 Stock-flow diagram of the goal seeking structure

The model equations are B.10 and B.11.

$$Adjustment\ flow = Adjustment\ fraction \times Discrepancy \qquad (B.10)$$

$$Discrepancy = Goal - Stock \qquad (B.11)$$

where "Goal" is a constant value and "Adjustment fraction" is a nonnegative constant value.

The diagram in Figure B.9 and equations B.10 and B.11 define a goal seeking structure. The simplified differential equation that corresponds to this structure is given in Equation B.12.

$$\frac{dStock}{dt} = Adjustment\ flow = Adjustment\ fraction \times (Goal - Stock) \qquad (B.12)$$

Block diagram of the goal seeking structure is given in Figure B.10.

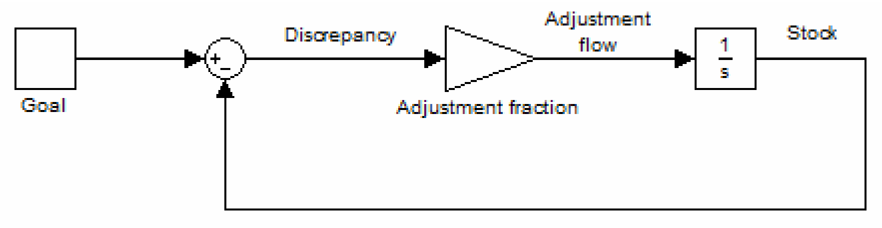

Figure B.10 Block diagram of the goal seeking structure

Capacitated growth structure (S-shaped growth caused by a capacity limit)

Stock-flow diagram of a capacitated growth structure is given in Figure B.11.

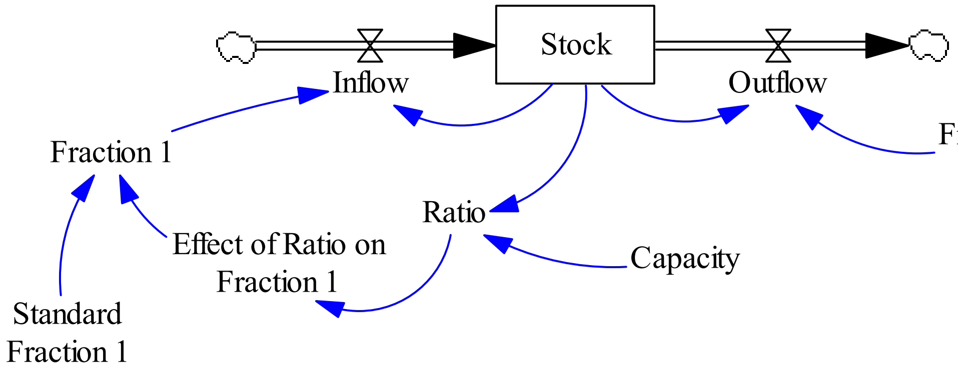

Figure B.11 Stock-flow diagram of the capacitated growth structure

The model equations are B.13-B.17.

$$Inflow = Fraction\ 1 \times Stock \tag{B.13}$$

$$Fraction\ 1 = Effect\ of\ \ Ratio\ on\ Fraction\ 1 \times Standard\ \ Fraction\ 1 \tag{B.14}$$

$$Effect\ of\ \ Ratio\ on\ Fraction\ 1 = f\left(Ratio\right) \tag{B.15}$$

$$Ratio = \frac{Stock}{Capacity} \tag{B.16}$$

$$Outflow = Fraction\ 2 \times Stock \tag{B.17}$$

The diagram in Figure B.11 and equations B.13-B.17 define a capacitated growth structure. The simplified differential equation that corresponds to this structure is given in Equation B.18.

$$\begin{aligned}\frac{dStock}{dt} &= Inflow - Outflow \\ &= f\left(\frac{Stock}{Capacity}\right) \times Standard\ Fraction\ 1 \times Stock - Fraction\ 2 \times Stock\end{aligned} \tag{B.18}$$

Block diagram of the capacitated growth structure is given in Figure B.12.

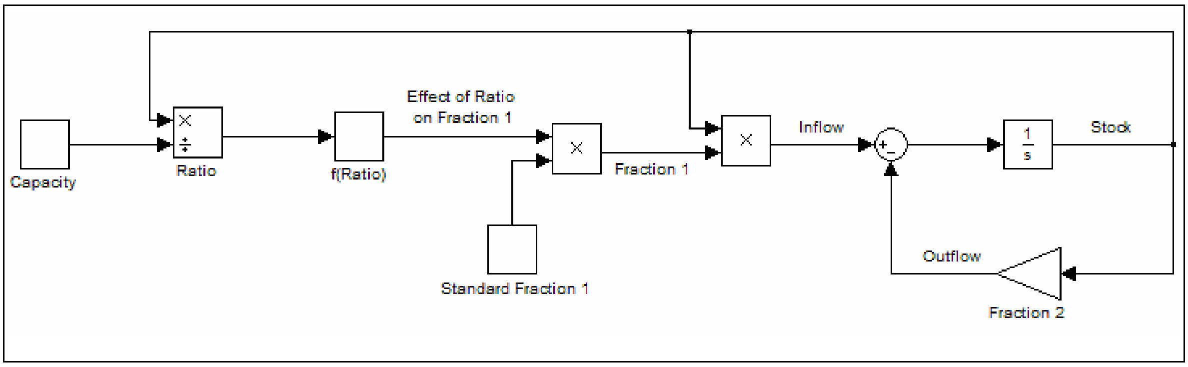

Figure B.12 Block diagram of the capacitated growth structure

As an example, assume that *f*(*Ratio*) is given by Equation B.19.

$$Effect\ of\ Ratio\ on\ Fraction\ 1 = f\left(Ratio\right) = 1 - 0.75 \times Ratio \tag{B.19}$$

The corresponding part of the block diagram representing *f*(*Ratio*), which is obtained under this assumption (Equation B.19), is given in Figure B.13.

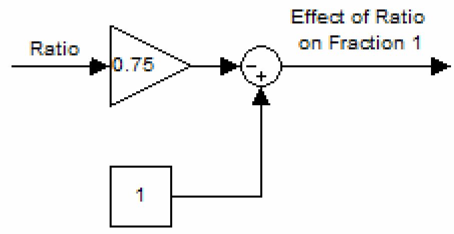

Figure B.13 Block diagram of an example *f*(*Ratio*)

Growth with overshoot structure (caused by a delayed effect of capacity limit)

Stock-flow diagram of a growth with overshoot structure is given in Figure B.14.

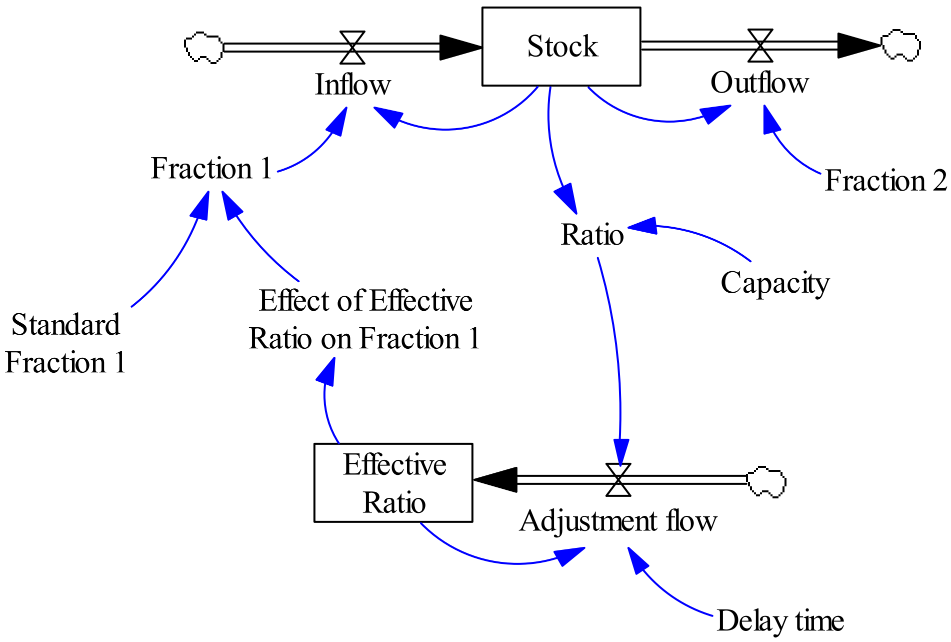

Figure B.14 Stock-flow diagram of the growth with overshoot structure

The model equations are B.20-B.25.

$$Inflow = Fraction\ 1 \times Stock \qquad \text{(B.20)}$$

$$Fraction\ 1 = Standard\ \ Fraction\ 1 \times Effect\ of\ \ Effective\ Ratio\ on\ Fraction\ 1 \qquad (B.21)$$

$$Effect\ of\ \ Effective\ Ratio\ on\ Fraction\ 1 = f\left(Effective\ Ratio\right) \qquad (B.22)$$

$$Adjustment\ \ flow = \frac{Ratio - Effective\ Ratio}{Delay\ time} \qquad (B.23)$$

$$Ratio = \frac{Stock}{Capacity} \qquad (B.24)$$

$$Outflow = Fraction\ \ 2 \times Stock \qquad (B.25)$$

The diagram in Figure B.14 and equations B.20-B.25 define a growth with overshoot structure. The simplified set of differential equations that corresponds to this structure is given in equation B.26 and B.27.

$$\begin{aligned} \frac{dStock}{dt} &= Inflow - Outflow \\ &= Standard\ \ Fraction\ 1 \times f\left(Effective\ Ratio\right) \times Stock - Fraction\ \ 2 \times Stock \end{aligned} \qquad (B.26)$$

$$\frac{dEffective\ Ratio}{dt} = Adjustment\ \ flow = \frac{\frac{Stock}{Capacity} - Effective\ Ratio}{Delay\ time} \qquad (B.27)$$

Block diagram of the growth with overshoot structure is given in Figure B.15.

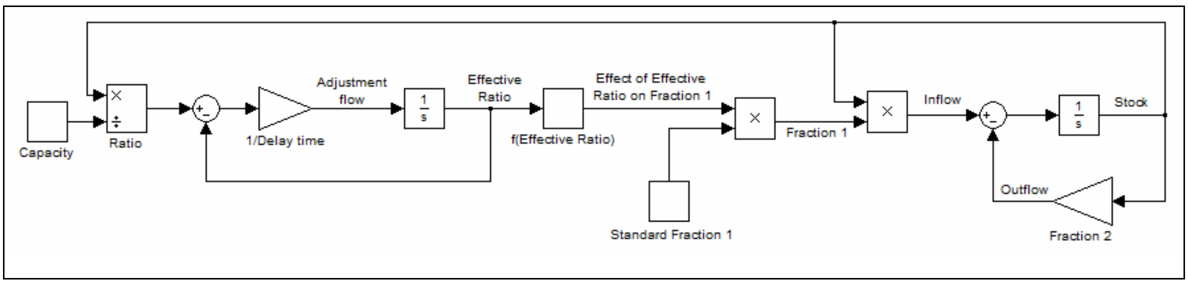

Figure B.15 Block diagram of the growth with overshoot structure

Continuous material delay structures

Stock-flow diagram of a first order continuous material delay structure is given in Figure B.16.

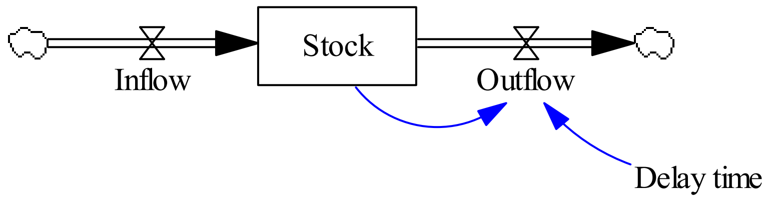

Figure B.16 Stock-flow diagram of the first order continuous material delay structure

Outflow equation of the model is given in Equation B.28.

$$Outflow = \frac{Stock}{Delay\ time} \tag{B.28}$$

The diagram in Figure B.16 and Equation B.28 define a first order continuous material delay structure. The simplified differential equation that corresponds to this structure is given in Equation B.29.

$$\frac{dStock}{dt} = Inflow - Outflow = Inflow - \frac{Stock}{Delay\ time} \tag{B.29}$$

where “Delay time” is a nonnegative constant value.

Block diagram of the first order continuous material delay structure is given in Figure B.17.

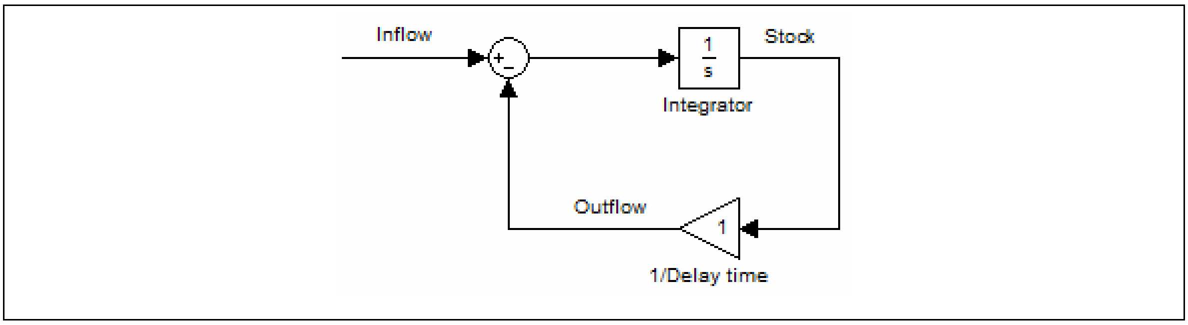

Figure B.17 Block diagram of the first order continuous material delay structure

Stock-flow diagram of a third order continuous material delay structure is given in Figure B.18.

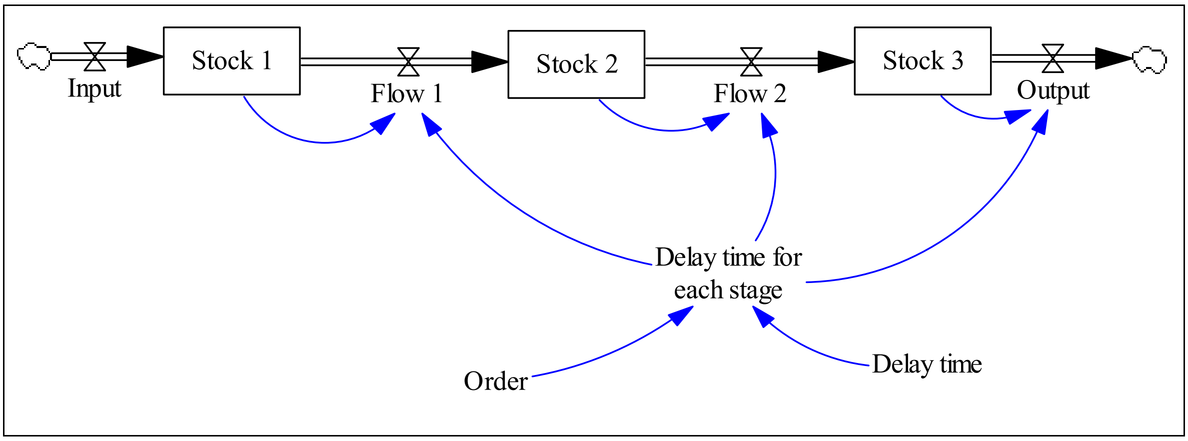

Figure B.18 Stock-flow diagram of the third order continuous material delay structure

The model equations are B.30-B.33.

$$Flow\,1 = \frac{Stock\ 1}{Delay\ time\ for\ each\ stage} \tag{B.30}$$

$$Flow\,2 = \frac{Stock\ 2}{Delay\ time\ for\ each\ stage} \tag{B.31}$$

$$Output = \frac{Stock\ 3}{Delay\ time\ for\ each\ stage} \tag{B.32}$$

$$\text{Delay time for each stage} = \frac{\text{Delay time}}{\text{Order}} \tag{B.33}$$

The diagram in Figure B.18 and equations B.30-B.33 define a third order continuous material delay structure. The simplified set of differential equations that corresponds to this structure is given in equations B.34, B.35, and B.36.

$$\frac{d\text{Stock 1}}{dt} = \text{Input} - \text{Flow1} = \text{Input} - \frac{\text{Stock 1}}{\text{Delay time}/\text{Order}} \tag{B.34}$$

$$\frac{d\text{Stock 2}}{dt} = \text{Flow1} - \text{Flow2} = \frac{\text{Stock 1}}{\text{Delay time}/\text{Order}} - \frac{\text{Stock 2}}{\text{Delay time}/\text{Order}} \tag{B.35}$$

$$\frac{d\text{Stock 3}}{dt} = \text{Flow2} - \text{Output} = \frac{\text{Stock 2}}{\text{Delay time}/\text{Order}} - \frac{\text{Stock 3}}{\text{Delay time}/\text{Order}} \tag{B.36}$$

where “Delay time” and “Order” are nonnegative constant values and “Order” corresponds to the number of state variables (i.e., stocks) in the material structure.

Block diagram of the third order continuous material delay structure is given in Figure B.19.

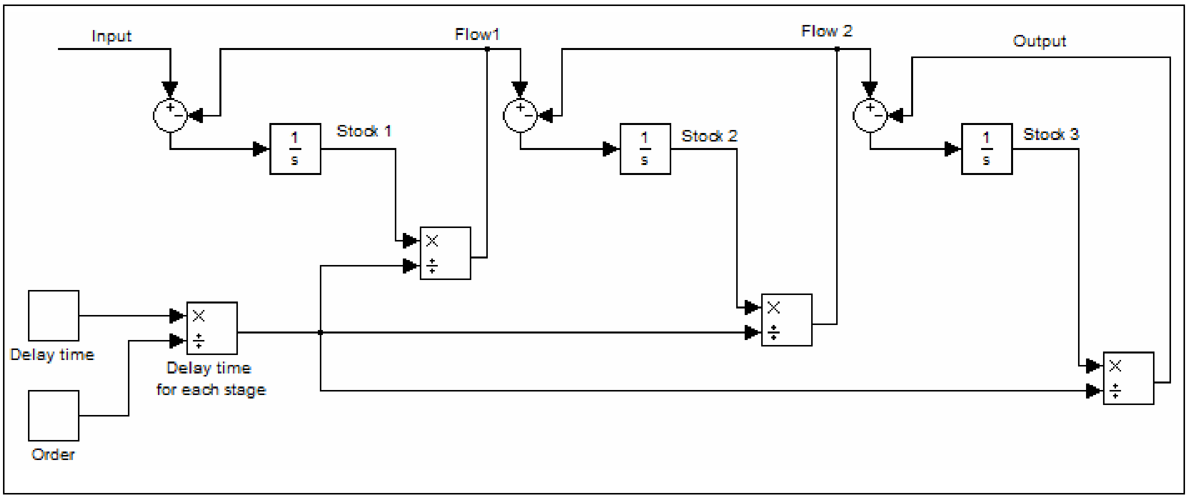

Figure B.19 Block diagram of the third order continuous material delay structure

Note that every material delay structure is an application of aforementioned draining structure. Hence, a material delay structure contains one or many draining structures.

Continuous information delay structures

Stock-flow diagram of a first order continuous information delay structure is given in Figure B.20.

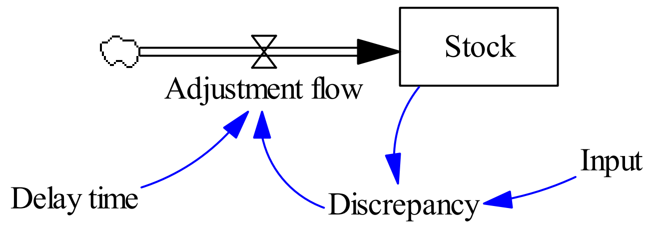

Figure B.20 Stock-flow diagram of the first order continuous information delay structure

Equations of the model are given in equations B.37 and B.38.

$$Adjustment\ flow = \frac{Discrepancy}{Delay\ time} \tag{B.37}$$

$$Discrepancy = Input - Stock \tag{B.38}$$

The diagram in Figure B.20 and equations B.37 and B.38 define a first order continuous information delay structure. The simplified differential equation that corresponds to this structure is given in Equation B.39.

$$\frac{dStock}{dt} = Adjustment\ flow = \frac{Input - Stock}{Delay\ time} \tag{B.39}$$

where “Delay time” is a nonnegative constant value.

Block diagram of the first order continuous information delay structure is given in Figure B.21.

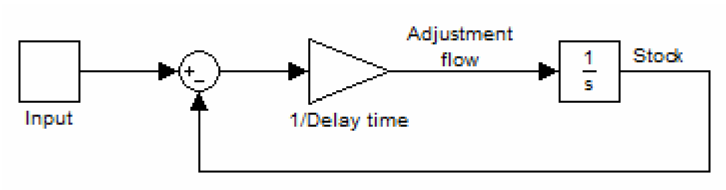

Figure B.21 Block diagram of the first order continuous information delay structure

Stock-flow diagram of a third order continuous information delay structure is given in Figure B.22.

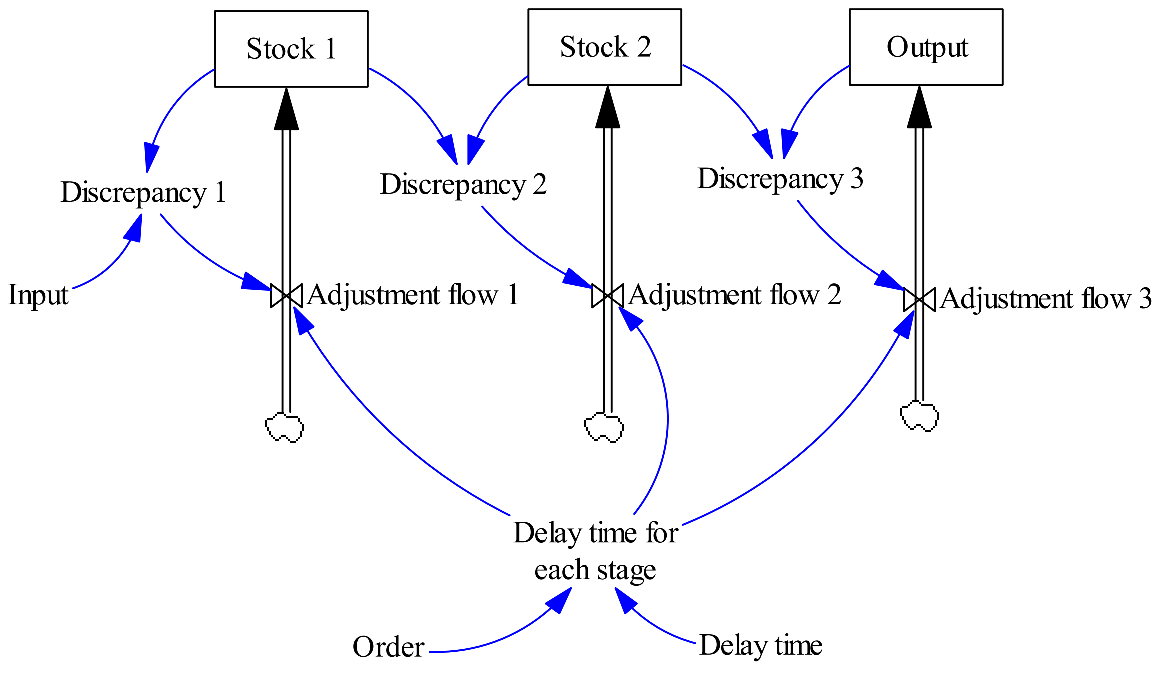

Figure B.22 Stock-flow diagram of the third order continuous information delay structure

The model equations are B.40-B.46.

$$Adjustment\ flow\,1 = \frac{Discrepancy\,1}{Delay\ time\ for\ each\ stage} \quad (B.40)$$

$$Adjustment\ flow\, 2 = \frac{Discrepancy\, 2}{Delay\ time\ for\ each\, stage} \tag{B.41}$$

$$Adjustment\ flow\, 3 = \frac{Discrepancy\, 3}{Delay\ time\ for\ each\, stage} \tag{B.42}$$

$$Discrepancy\, 1 = Input - Stock\ 1 \tag{B.43}$$

$$Discrepancy\ 2 = Stock\ 1 - Stock\ 2 \tag{B.44}$$

$$Discrepancy\, 3 = Stock\ 2 - Output \tag{B.45}$$

$$Delay\ time\ for\ each\, stage = \frac{Delay\ time}{Order} \tag{B.46}$$

The diagram in Figure B.22 and equations B.40-B.46 define a third order continuous information delay structure. The simplified differential equations that correspond to this structure are given in equations B.47, B.48, and B.49.

$$\frac{dStock\ 1}{dt} = Adjustment\ flow\, 1 = \frac{Input - Stock\ 1}{Delay\ time / Order} \tag{B.47}$$

$$\frac{dStock\ 2}{dt} = Adjustment\ flow\, 2 = \frac{Stock\ 1 - Stock\ 2}{Delay\ time / Order} \tag{B.48}$$

$$\frac{dOutput}{dt} = Adjustment\ flow\, 3 = \frac{Stock\ 2 - Output}{Delay\ time / Order} \tag{B.49}$$

where “Delay time” and “Order” are nonnegative constant values.

Block diagram of the third order continuous information delay structure is given in Figure B.23.

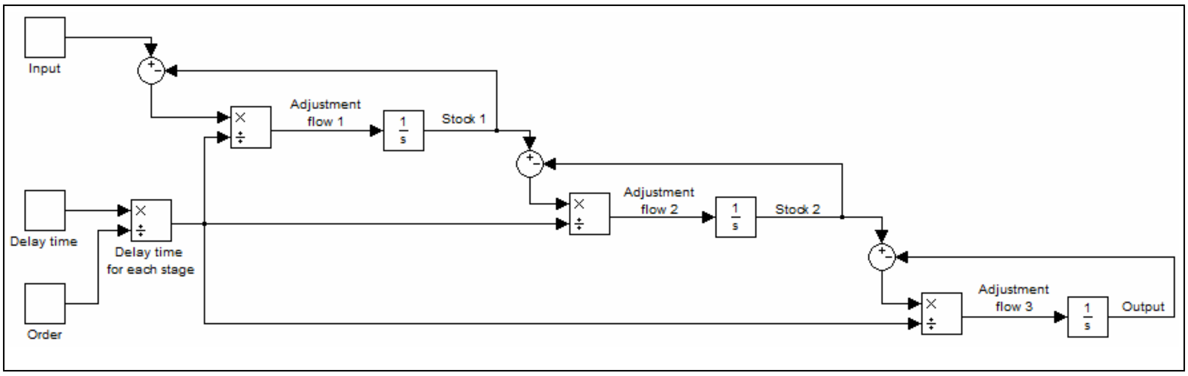

Figure B.23 Block diagram of the third order continuous information delay structure

Note that every information delay structure is an application of aforementioned goal seeking structure. Hence, an information delay structure contains one or many goal seeking structures.

Discrete material delay structure (pure delay)

Stock-flow diagram of a discrete material delay structure is given in Figure B.24.

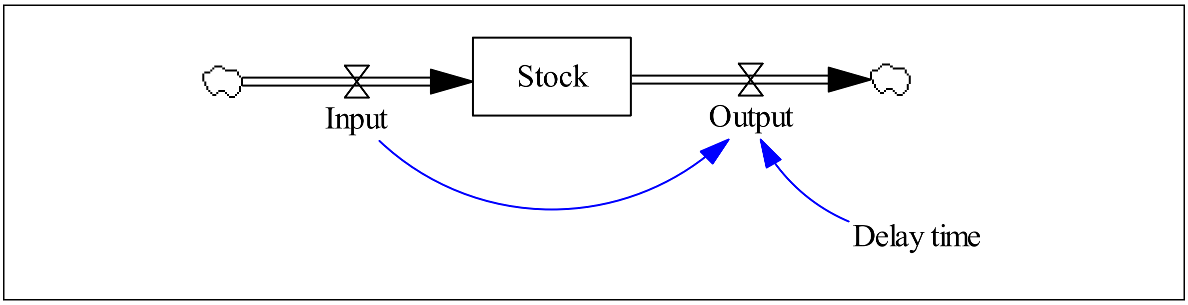

Figure B.24 Stock-flow diagram of the discrete material delay structure

Output equation of the model is given in Equation B.50.

$$Output(t) = \begin{cases} \dfrac{Stock(0)}{Delay\ time} & \text{for} \quad 0 \le t < Delay\ time \\ Input(t - Delay\ time) & \text{for} \quad t \ge Delay\ time \end{cases} \qquad (B.50)$$

where “Delay time” is a nonnegative constant value.

The diagram in Figure B.24 and Equation B.50 define an infinite order (i.e., discrete) material delay structure. The simplified differential equation that corresponds to this structure is given in Equation B.51.

$$\frac{dStock}{dt} = Input - Output$$
$$= \begin{cases} Input(t) - \dfrac{Stock(0)}{Delay\ time} & \text{for} \quad 0 \le t < Delay\ time \\ Input(t) - Input(t - Delay\ time) & \text{for} \quad t \ge Delay\ time \end{cases} \qquad \text{(B.51)}$$

Block diagram of the discrete material delay structure is given in Figure B.25.

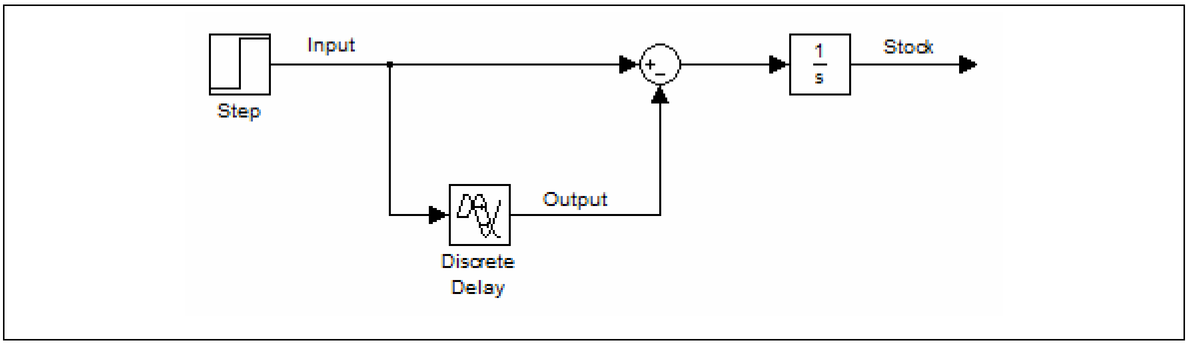

Figure B.25 Block diagram of the discrete material delay structure

Discrete information delay structure (pure delay)

Stock-flow diagram of a discrete information delay structure is given in Figure B.26.

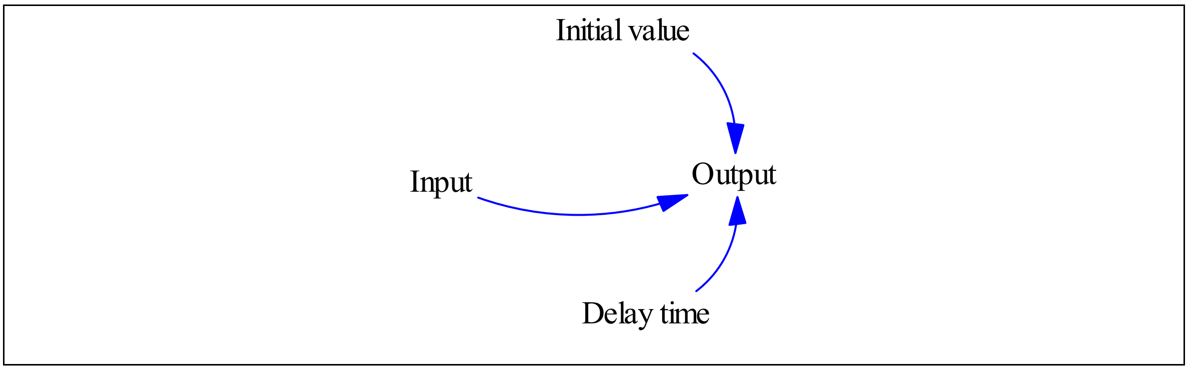

Figure B.26 Stock-flow diagram of the discrete information delay structure

Output equation of the model is given in Equation B.52.

$$Output(t) = \begin{Bmatrix} Initial\ value & \text{for} & 0 \le t < Delay\ time \\ Input(t - Delay\ time) & \text{for} & t \ge Delay\ time \end{Bmatrix} \quad \text{(B.52)}$$

where "Delay time" is a nonnegative constant value.

The diagram in Figure B.24 and Equation B.50 define an infinite order (i.e., discrete) information delay structure. Note that there is no simplified differential equation that corresponds to this structure as there is no stock in Figure B.24.

Block diagram of the discrete information delay structure is given in Figure B.27.

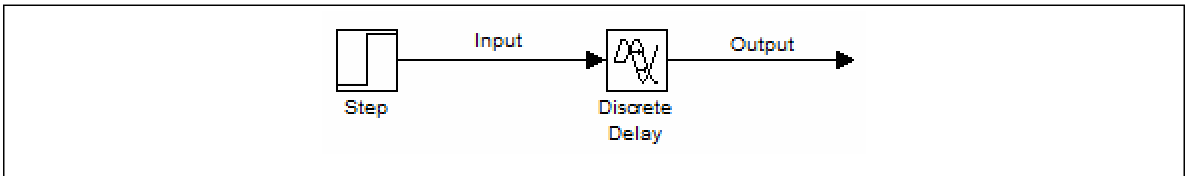

Figure B.27 Block diagram of the discrete information delay structure

Oscillating structure

Stock-flow diagram of an oscillating structure is given in Figure B.28.

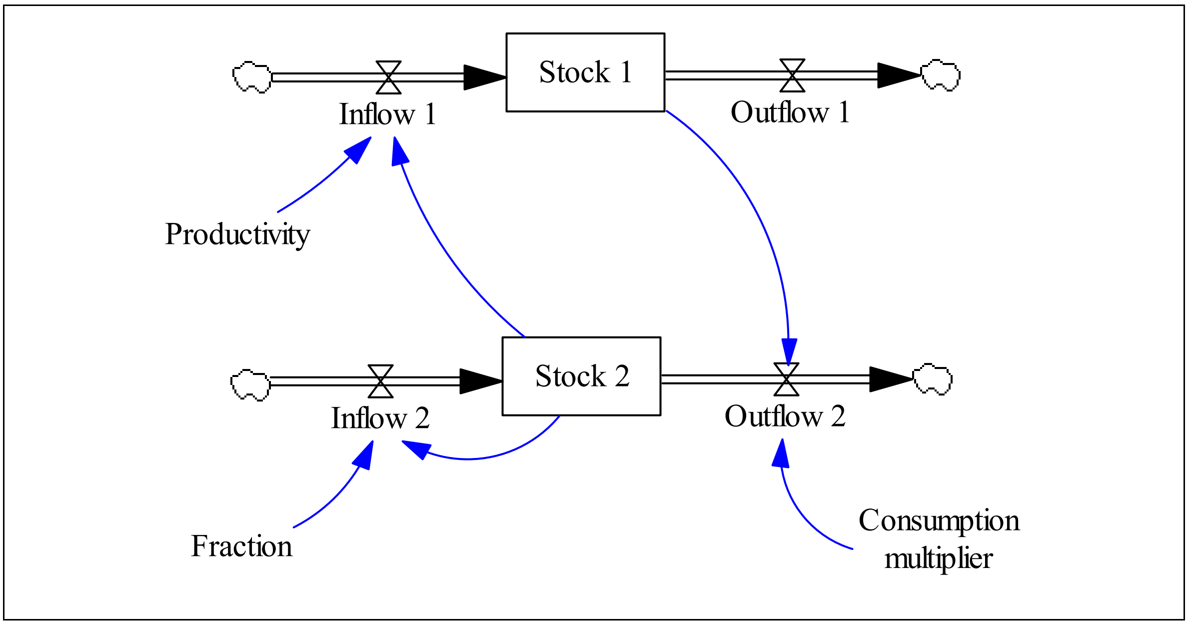

Figure B.28 Stock-flow diagram of the oscillating structure

The model equations are B.53, B.54, and B.55.

$$Inflow\ 1 = Productivity \times Stock\ 2 \tag{B.53}$$

$$Inflow\ 2 = Fraction \times Stock\ 2 \tag{B.54}$$

$$Outflow\ 2 = Consumption\ multiplier \times Stock\ 1 \tag{B.55}$$

The diagram in Figure B.28 and equations B.53, B.54, and B.55 define an oscillating structure. The simplified differential equations that correspond to this structure are given in equations B.56 and B.57.

$$\frac{dStock\ 1}{dt} = Inflow\ 1 - Outflow\ 1 = Productivity \times Stock\ 2 - Outflow\ 1 \tag{B.56}$$

$$\frac{dStock\ 2}{dt} = Inflow\ 2 - Outflow\ 2$$
$$= Fraction \times Stock\ 2 - Consumption\ multiplier \times Stock\ 1 \qquad (B.57)$$

where "Productivity" and "Consumption multiplier" are nonnegative constant values and "Fraction" is a constant value.

Block diagram of the oscillating structure is given in Figure B.29.

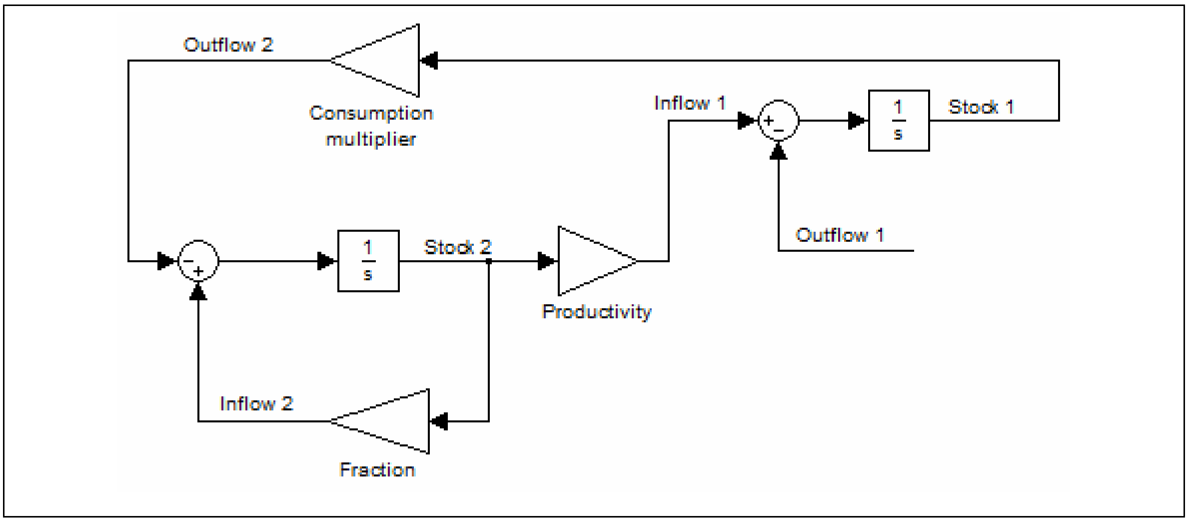

Figure B.29 Block diagram of the oscillating structure

Simple goal setting structure

Stock-flow diagram of the simple goal setting structure is given in Figure B.30.

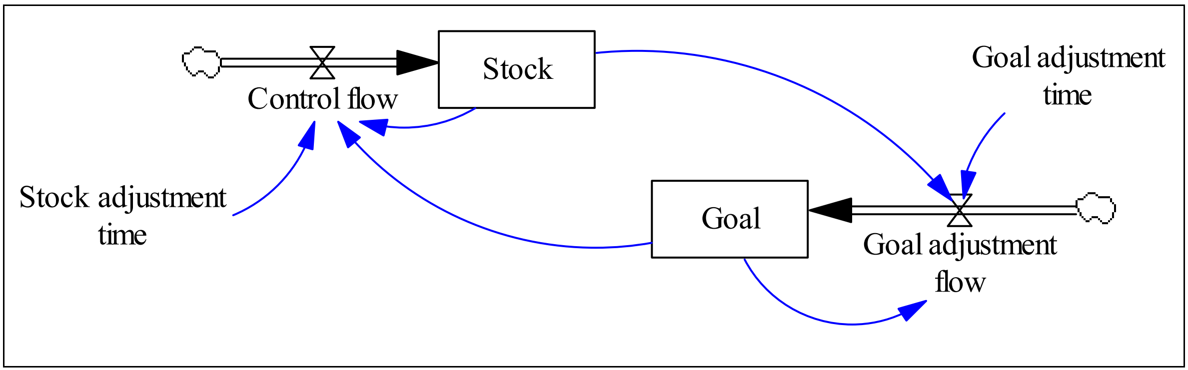

Figure B.30 Stock-flow diagram of the simple goal setting structure

The model equations are B.58 and B.59.

$$Control\ flow = \frac{Goal - Stock}{Stock\ adjustment\ time} \tag{B.58}$$

$$Goal\ adjustment\ flow = \frac{Stock - Goal}{Goal\ adjustment\ time} \tag{B.59}$$

The diagram in Figure B.30 and equations B.58 and B.59 define the simple goal setting structure. The simplified set of differential equations that correspond to this structure are given in equations B.60 and B.61.

$$\frac{dStock}{dt} = Control\ flow = \frac{Goal - Stock}{Stock\ adjustment\ time} \tag{B.60}$$

$$\frac{dGoal}{dt} = Goal\ adjustment\ flow = \frac{Stock - Goal}{Goal\ adjustment\ time} \tag{B.61}$$

where "Stock adjustment time" and "Goal adjustment time" are nonnegative constant values.

Block diagram of the simple goal setting structure is given in Figure B.31.

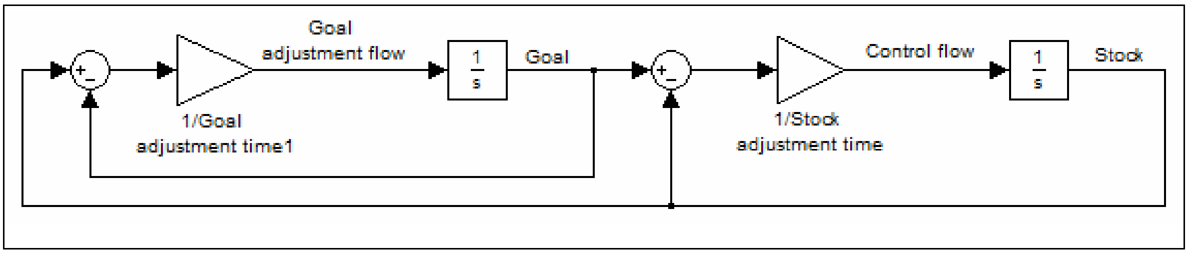

Figure B.31 Block diagram of the simple goal setting structure

Epidemic model structure

Stock-flow diagram of the epidemic model structure is given in Figure B.32.

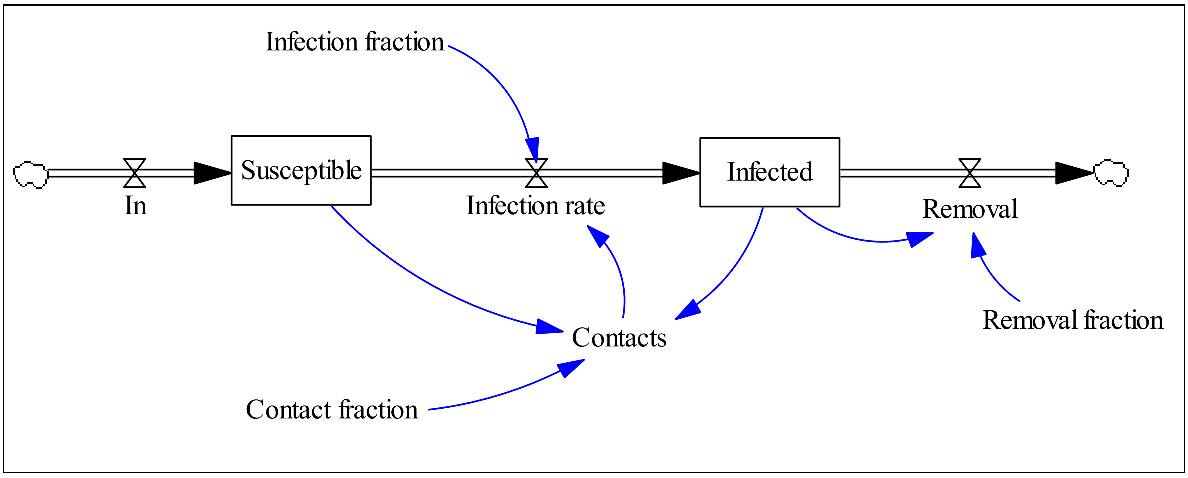

Figure B.32 Stock-flow diagram of the epidemic model structure

Model equations are B.62, B.63, and B.64.

$$Infection\ rate = if \times Contacts \tag{B.62}$$

$$Contacts = cf \times S \times I \qquad (B.63)$$

$$Removal = rf \times I \qquad (B.64)$$

where *if* stands for “Infection fraction”, *cf* stands for “Contact fraction”, and *rf* stands for “Removal fraction”. *S* and *I* stand, respectively, for “Susceptible” and “Infected”.

The diagram in Figure B.32 and equations B.62, B.63, and B.64 define the epidemic model structure. The simplified set of differential equations that correspond to this structure are given in equations B.65 and B.66.

$$\frac{dS}{dt} = In - Infection\ rate = In - if \times cf \times S \times I \qquad (B.65)$$

$$\frac{dI}{dt} = Infection\ rate - Removal = if \times cf \times S \times I - rf \times I \qquad (B.66)$$

where *if*, *cf*, and *rf* are nonnegative constant values.

Block diagram of the epidemic model structure is given in Figure B.33.

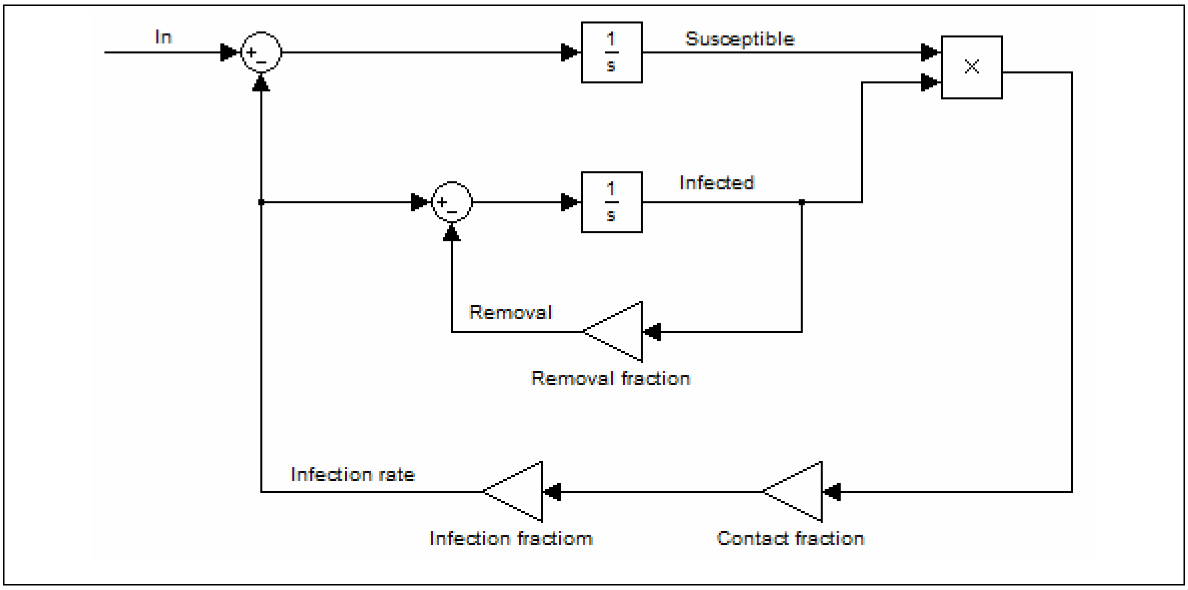

Figure B.33 Block diagram of the epidemic model structure

Stock management with a first order supply line delay structure

Stock-flow diagram of a stock management structure with a first order supply line delay is given in Figure B.34.

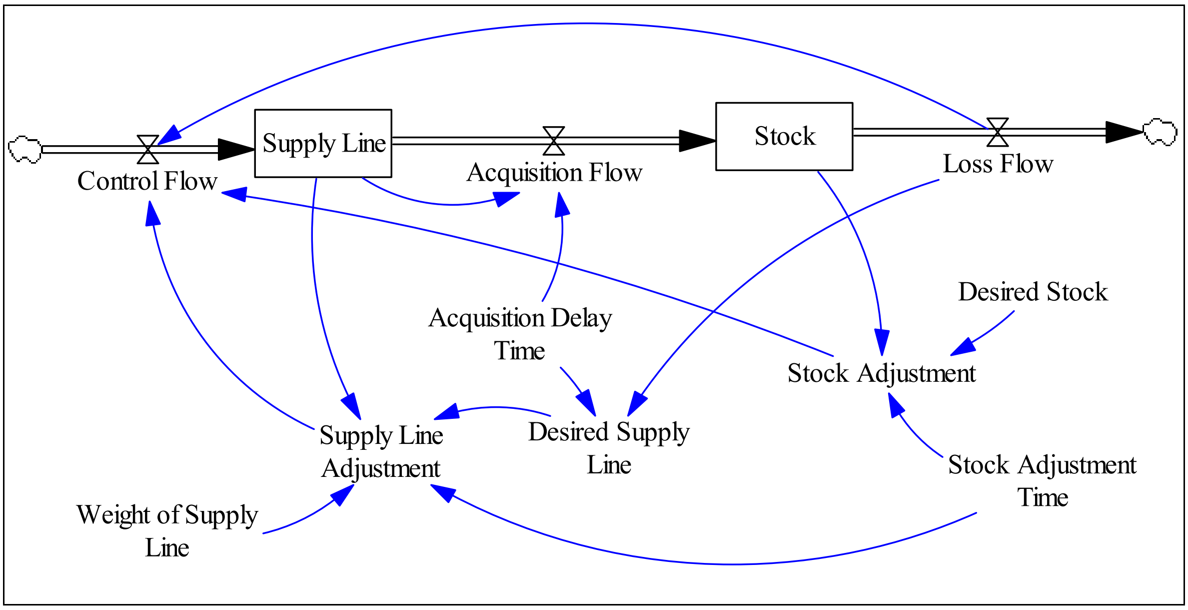

Figure B.34 Stock-flow diagram of the stock management structure with a first order supply line delay

The model equations are B.67-B.71.

$$CF = LF + SA + SLA \tag{B.67}$$

$$SA = \frac{S^* - S}{sat} \tag{B.68}$$

$$SLA = wsl \times \frac{SL^* - SL}{sat} \tag{B.69}$$

$$AF = \frac{SL}{adt} \tag{B.70}$$

$$SL^* = adt \times LF \tag{B.71}$$

where *CF* stands for "Control Flow", *LF* stands for "Loss Flow", *SA* stands for "Stock Adjustment", *SLA* stands for "Supply Line Adjustment", *S** stands for "Desired Stock", *S* stands for "Stock", *sat* stands for "Stock Adjustment Time", *wsl* stands for "Weight of Supply Line", *SL** stands for "Desired Supply Line", *SL* stands for "Supply Line", *AF* stands for "Acquisition Flow", *adt* stands for "Acquisition Delay Time".

The diagram in B.34 and equations B.67-B.71 define a stock management structure with a first order supply line delay. The simplified set of differential equation that corresponds to this structure is given in equations B.72 and B.73.

$$\frac{dS}{dt} = AF - LF = \frac{SL}{adt} - LF \tag{B.72}$$

$$\frac{dSL}{dt} = CF - AF = LF + \frac{S^* - S}{sat} + wsl \times \frac{adt \times LF - SL}{sat} - \frac{SL}{adt} \tag{B.73}$$

Block diagram of the stock management structure with a first order supply line delay is given in Figure B.35.

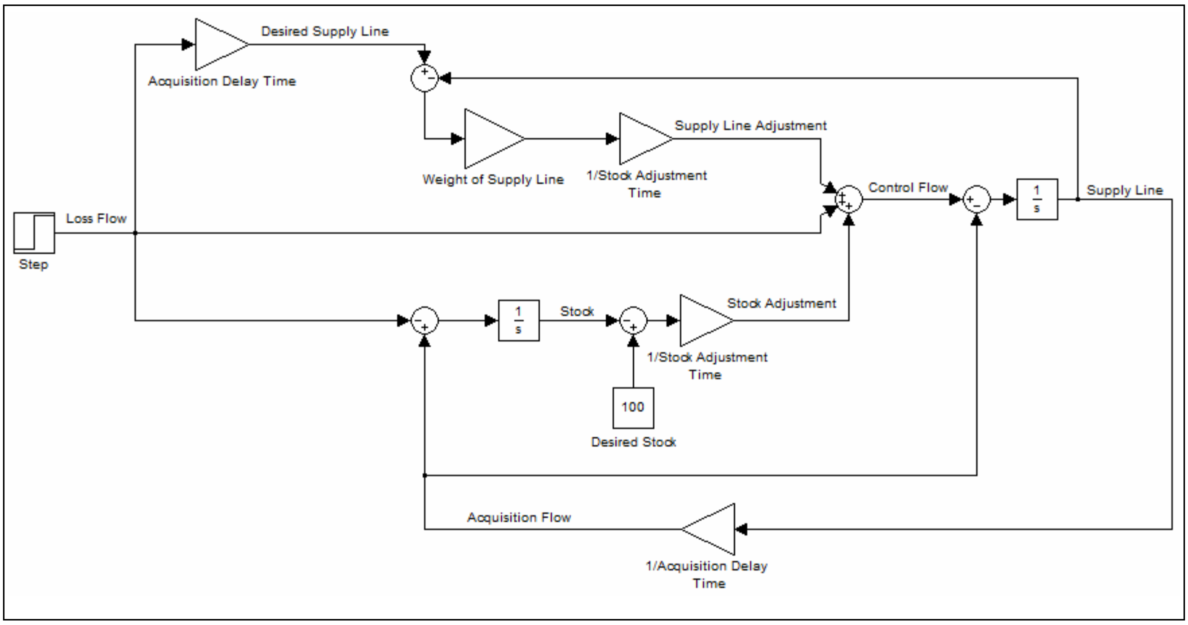

Figure B.35 Block diagram of the stock management structure with a first order supply line delay

Stock management with a third order supply line delay structure

Stock-flow diagram of a stock management structure with a third order supply line delay is given in Figure B.36.

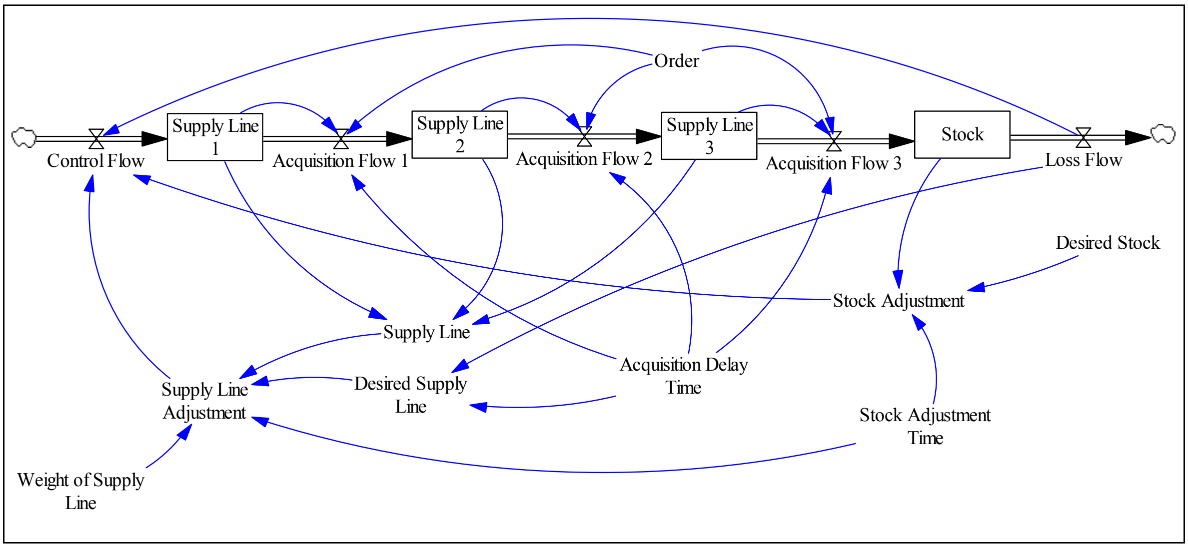

Figure B.36 Stock-flow diagram of the stock management structure with a third order supply line delay

The model equations are B.74-B.81.

$$CF = LF + SA + SLA \quad \text{(B.74)}$$

$$SA = \frac{S^* - S}{sat} \quad \text{(B.75)}$$

$$SLA = wsl \times \frac{SL^* - SL}{sat} \quad \text{(B.76)}$$

$$SL^* = adt \times LF \quad \text{(B.77)}$$

$$SL = SL1 + SL2 + SL3 \quad \text{(B.78)}$$

$$AF1 = \frac{SL1}{adt/Order} \tag{B.79}$$

$$AF2 = \frac{SL2}{adt/Order} \tag{B.80}$$

$$AF3 = \frac{SL3}{adt/Order} \tag{B.81}$$

where *CF* stands for "Control Flow", *LF* stands for "Loss Flow", *SA* stands for "Stock Adjustment", *SLA* stands for "Supply Line Adjustment", *S** stands for "Desired Stock", *S* stands for "Stock", *sat* stands for "Stock Adjustment Time", *wsl* stands for "Weight of Supply Line", *SL** stands for "Desired Supply Line", *SL* stands for "Supply Line", *adt* stands for "Acquisition Delay Time", *SL1* stands for "Supply Line 1", *SL2* stands for "Supply Line 2", *SL3* stands for "Supply Line 3", *AF1* stands for "Acquisition Flow 1", *AF2* stands for "Acquisition Flow 2", *AF3* stands for "Acquisition Flow 3".

The diagram in B.36 and equations B.74-B.81 define a stock management structure with a third order supply line delay. The simplified set of differential equation that corresponds to this structure is given in equations B.82, B.83, B.84, and B.85.

$$\frac{dS}{dt} = AF3 - LF = \frac{SL3}{adt/Order} - LF \tag{B.82}$$

$$\frac{dSL1}{dt} = CF - AF1 = LF + \frac{S^* - S}{sat} + wsl \times \frac{adt \times LF - SL}{sat} - \frac{SL1}{adt/Order} \tag{B.83}$$

$$\frac{dSL2}{dt} = AF1 - AF2 = \frac{SL1}{adt/Order} - \frac{SL2}{adt/Order} \tag{B.84}$$

$$\frac{dSL3}{dt} = AF2 - AF3 = \frac{SL2}{adt/Order} - \frac{SL3}{adt/Order} \tag{B.85}$$

Block diagram of the stock management structure with a third order supply line delay is given in Figure B.37.

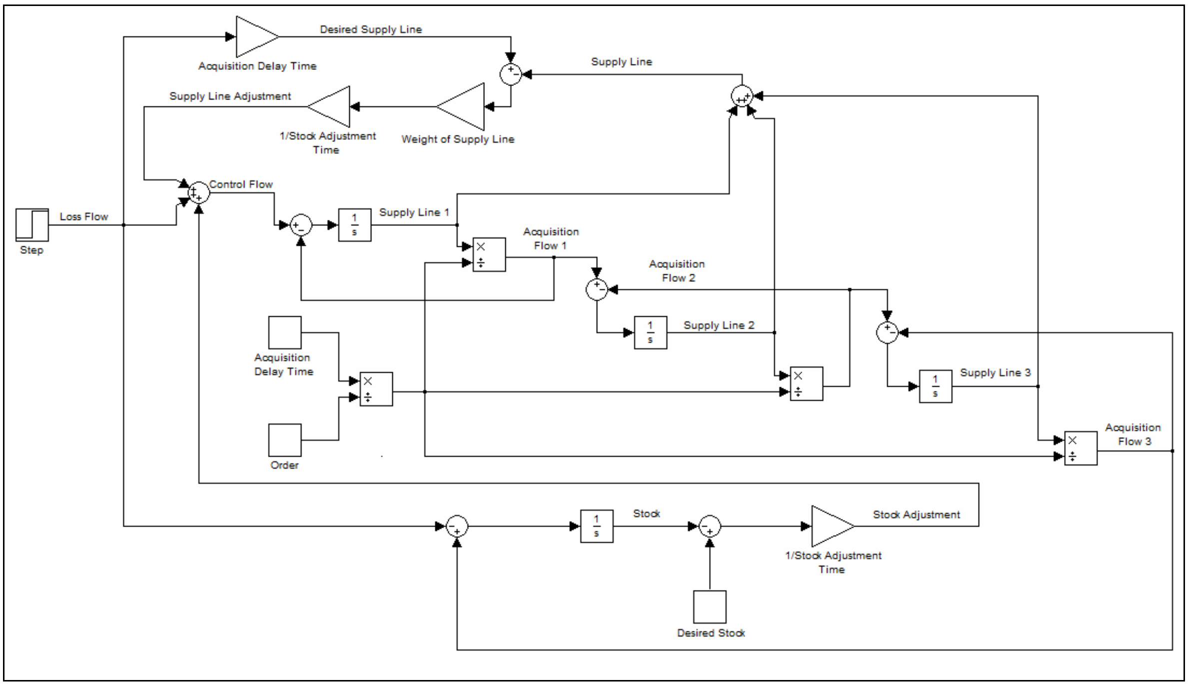

Figure B.37 Stock-flow diagram of the stock management structure with a third order supply line delay