AN ANALYTICAL FORM FOR TABLE FUNCTIONS USED IN SYSTEM DYNAMICS

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ABSTRACT

Table functions are frequently used in System Dynamics. An analytical function capable of representing a large class of such functions is defined and discussed.

1. INTRODUCTION

In modelling, a multiplier function, f(t), having the following essential properties is often needed:

(a) f(t) is defined for $0 < t < \infty$

(b) f(1) = 1

(c) f(t) must approach a specified horizontal asymptote A (> 1) as $t \to \infty$

(d) f(t) must approach a given value B (0 < B < 1) as $t \to 0$.

Such a function can be defined by specifying a number of points through which its graph must pass. Linear interpolation, for example, can then be used to generate other points. This yields what is known as a table function[1].

We define a continuous function f(A, B, g, t), having the properties (a) to (d) above.

f (A, B, g, t) is completely determined by the parameters A, B and g. A and B have the meanings defined above and are supposed to be supplied. g is the gradient of f (A, B, g, t) at t = 1. (g must also be given). t is the independent variable.

Since only three parameters are involved, specification and subsequent modification or adjustment of the multiplier function are readily accomplished. Further, such a function is readily amenable to sensitivity analysis and facilitates an investigation of stability in a feedback situation.

2. THE FUNCTION f (A, B, g, t)

Given $A > 1$, $0 < B < 1$ and $g > 0$

$$\text{let} \quad E = (A/B) - 1 \quad \ldots\ldots \quad (1)$$

$$C = \ln [E/(A - 1)] \quad \ldots\ldots \quad (2)$$

$$\text{and} \quad s = g.A/ [(A - 1).c] \quad \ldots\ldots \quad (3)$$

$$\text{Define } f(A, B, g, t) = A/[1 + E.\exp(-C.t^s)] \quad \ldots\ldots \quad (4)$$

3. VERIFICATION OF THE PROPERTIES

(a), (b): Properties (a) and (b) can be verified immediately.

Other properties can be established after $\frac{d}{dt} f(A, B, g, t)$ is determined:

$$\frac{d}{dt} f(A, B, g, t) = -F.[-E.C.s.t^{s-1}.\exp(-C.t^s)] \quad \ldots \quad (5)$$

$$\text{where } F = A/[1 + E.\exp(-C.t^s)]^2 \quad \ldots\ldots \quad (6)$$

Using (2), (3), (4), (6) and property (b) we immediately obtain $\frac{d}{dt} f(A, B, g, 1) = g$ confirming that g is in fact the gradient of f (A, B, g, t) at t = 1.

(c): The fact that $f(A, B, g, t) \to A$ as $t \to \infty$ follows directly from (4).

Since, by (6), $F \to A$ as $t \to \infty$ while in (5) the term $t^{s-1} \exp.(-C\, t^s) \to 0$ as $t \to \infty$, it follows from (5) that $\frac{d}{dt} f(A, B, g, t) \to 0$ as $t \to \infty$. Thus it is apparent that f (A, B, g, t) approaches the value A asymptotically.

(d): Using (4) and (1) it can easily be shown that $f(A, B, g, t) \to B$ as $t \to 0$.

The manner of this approach can be ascertained by considering (5) again, together with (6):

If $s > 1$ it is clear that $\frac{d}{dt} f(A, B, g, t) \to 0$ as $t \to 0$ so that f (A, B, g, t) approaches the value B asymptotically. (See figure (a) for examples).

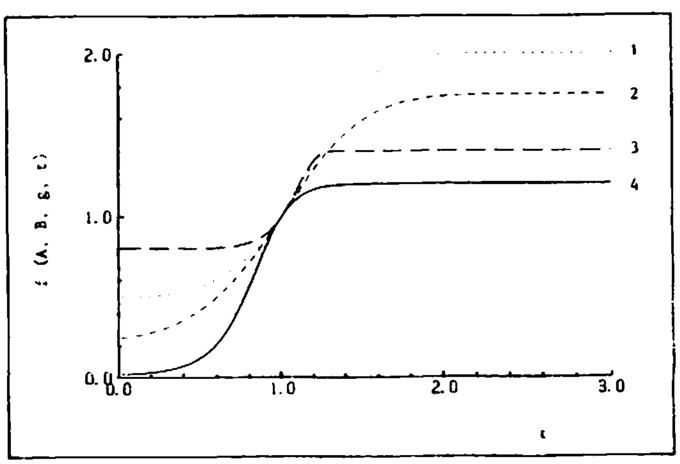

Figure a: Constant g, s>1.

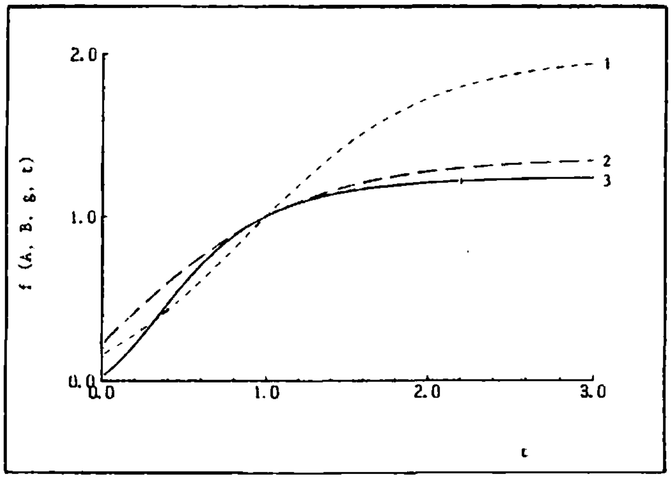

Figure b: s ≦ 1, various g.

If s < 1, on the other hand, consideration of $\frac{d^2}{dt^2}$ f (A, B, g, t) shows that $\frac{d}{dt}$ f (A, B, g, t) does not necessarily tend to ∞ as t = 0. This means that the vertical axis through t = 0 does not necessarily act as a vertical asymptote for the function (see figure (b) for examples).

If s = 1 neither of the above considerations apply.

In all cases however, $\frac{d}{dt}$f (A, B, g, t) is never zero.

4. ALTERNATIVE FORM

It is apparent from the above discussion that the index s is a better indicator of the shape of the graph in the region 0 < t < 1 than is g.

Since the behaviour of the function as t → 0 may well be a more important consideration than the precise gradient of function at t = 1, we give an alternative but equivalent definition of f in which the parameter s is given priority:

Given A > 1, 0 < B < 1 and s > 0,

$$f(A, B, s, t) = A/[1 + E.\exp(-C.t^s)] \quad \text{.......... (7)}$$

where as before

$$E = (A/B) - 1 \quad \text{.......... (8)}$$

$$\text{and } C = \ln[E/(A-1)] \quad \text{.......... (9)}$$

$$\text{The transformation } g = s.C.(A-1)/A \quad \text{.......... (10)}$$

gives the gradient of f at t = 1.

5. EXAMPLES

Graphs of various representative functions f (A, B, g,t) are presented in figures (a) and (b). In figure (a) s > 1 in every case whereas in figure (b) s ≦ 1 in every case.

Figures (c) and (d) show graphs of various functions expressed in the alternative form f (A, B, s, t). In (c), the effect of varying s is illustrated while in (d) s is the same in all cases. Table 1 shows the parameter values used for these graphs.

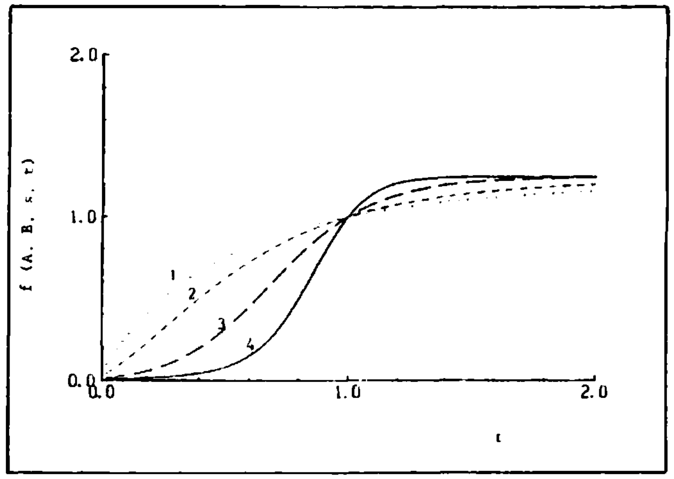

Figure c: Various s.

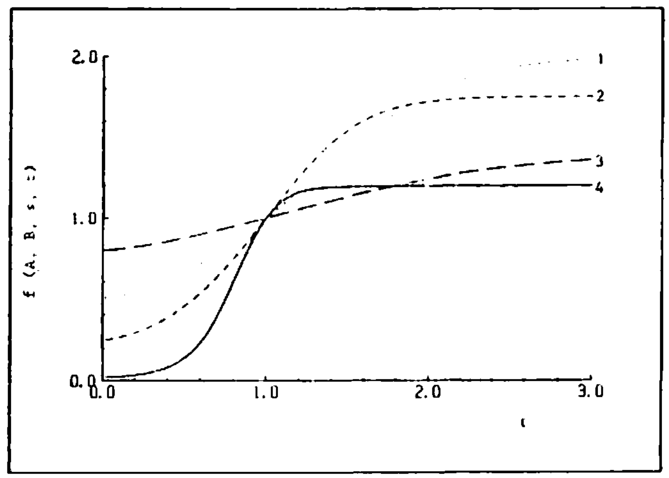

Figure d: Constant s.

6. APPLICATION

As an illustration of an application we quote an example from a System Dynamics model of a rural cattle herd[2].

Since the herd is subject to minimal management and the animals rely entirely on foraging, death rates and fecundity rates vary according to forage availability.

The normal values of these rates are known but these need to be modified after a time delay by suitable multiplier functions to allow for variation depending on forage availability. (Details of the time delay mechanism incorporated are omitted here).

Extreme values for these variations were deduced by referring to data from Agricultural Experimental Stations. These extremes should be approached asymptotically. Thus multiplier functions of the type illustrated in figure (a) were considered suitable. It remained to establish values of the parameter g in each case. This was accomplished by adjusting the values of g in each case until the average values of the death rates and decundity rates as predicted by the model in computer simulations were all sufficiently close (within 5%) to the actual rates as in the known data.

	Figure a				Figure b				Figure c				Figure d			
Graph	A	B	g	s	A	B	g	s	A	B	s	g	A	B	s	g
1	2.00	0.50	1.5	2.73	2.00	0.15	1.0	0.80	1.25	0.01	0.20	0.25	2.00	0.50	1.5	0.82
2	1.75	0.25	1.5	1.68	1.37	0.20	0.5	0.67	1.25	0.01	0.30	0.37	1.75	0.25	1.5	1.34
3	1.40	0.80	1.5	8.35	1.25	0.01	0.5	0.40	1.25	0.01	0.60	0.75	1.40	0.80	1.5	0.27
4	1.20	0.20	1.5	1.58	–				1.25	0.01	1.25	1.55	1.20	0.02	1.5	1.42

Table 1: Parameter values. Note particularly that B must be greater than 0 (see equation 1) and A must be greater than 1 (see equation 2).

7. CONCLUSION

The great ease with which these functions can be dealt with is illustrated by the fact that the parameters for all the graphs in this paper were input to an H-P micro-processor and the graphs produced by the plotter in under 15 minutes.

In the model referred to in 6-Application, a total of seven such functions is used. Calibration of the model would have been cumbersome indeed if these functions had been specified in table form.

It seems to the author that the alternative form of the function is particularly convenient since the shape of the graph in the region $0 < t < 1$ can then be easily determined (see figure (c)).

On the other hand, if the gradient of the function near (1,1) is fixed then it is necessary to use the original form of the function. (Compare figures (d) and (a)).

In any case, it is a simple matter to adjust individually the positions of the asymptotes of a particular function without interfering with the behaviour of the function near (1,1) (see figure (a)), or alternatively, to modify the function near (1,1) without altering the behaviour of the function for extreme values of t. (see figure (c)).

REFERENCES

1. ALFELD, L.E. and GRAHAM, A.K., (1976), Growth and Equilibrium in a Finite Area: The BSNSS2 Model, Chapter 2 in *Introduction to Urban Dynamics*, Wright-Allen Press, Cambridge, Massachusetts.

2. UYS, P.W., HEARNE, J.W., and COLVIN, P., A Model for estimating the Beef Production Potential of a cattle herd in a rural Sweetveld region – submitted to *Agricultural Systems*.