Stability, Feedback, and Delays

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Measuring several components of the dynamics and feeding it back to the system to alter its behaviour is a key concept of system theory. More often than not, this process is not instantaneous but involves some time delay, either in the measurement or feedback channels, or in the processing of the data. Thus, the information fed back to the system is based on the state at some past instant of time. In most engineering problems such delays can be neglected without introducing too much error; in other areas involving biological, physiological, or large scale socio-economic systems, the delays can be large and their presence can change the dynamics of the system significantly. Delays in the feedback path are generally looked upon as undesirable from a stability the point of view. In this work, some positive aspects of delays are presented. It is shown that a judicious choice of the feedback law in the presence of delays can be used to stabilize equilibria.

It is known that delays can have the undesirable effect of destabilizing systems; examples abound in the literature. For instance, Diekmann et al. (1995) give a detailed analysis for the first order linear system under delayed feedback

$$\dot{x}(t) + bx(t) = cx(t-1),$$

determining regions of stability and instability in the b–c parameter plane. Similarly, Cooke and Grossman (1982) analyse stability changes in several systems as the delay is varied. Suh and Bien (1979; 1980) give numerical results on the positive uses of delays in feedback, altough without addressing the stability issues. So far, most of the efforts in the context of feedback delays appears to have gone to achieve stability *despite* the delay. Here we give an example where stability can be obtained only *because of* the delay.

We consider a second order system governed by the differential equation

$$\ddot{x}(t) + \omega^2 x(t) = 0,$$

whose solutions are periodic with frequency ω. The aim is to quench out the oscillations and make the origin asymptotically stable by measuring and feeding back the values of x. If the total delay involved in this process is denoted by τ, then the dynamics of the system is given by

$$\ddot{x}(t) + \omega^2 x(t) = cx(t - \tau), \qquad \tau > 0, \tag{1}$$

where c is a constant. Note that there is no loss of generality in taking either the system or the feedback law as linear, since stability of a (hyperbolic) equilibrium will be determined by the linear part only. The question is then that given the values of ω and τ, if it is possible to choose c such that the equilibrium point $x = 0$ of the system (1) is asymptotically stable. Our main result gives a positive answer.

Theorem *If ω and τ are such that $\omega\tau \neq n\pi$ for any integer n, then there is a real number c such that the origin of equation (1) is asymptotically stable.*

Proof We briefly outline the proof. The characteristic equation of the system is found, by making the ansatz $x(t) = \exp(\lambda t)$ as usual, to be

$$\Delta(\lambda) = \lambda^2 + \omega^2 - c\exp(-\lambda\tau) = 0. \tag{2}$$

Asymptotic stability is obtained when every root λ of Δ has a negative real part. Now, Δ has an infinite number of roots, reflecting the fact that the presence of the delay gives rise to an infinite-dimensional dynamical system. Thus, it is impossible even numerically to check the real part of every root, and one has to use indirect methods. One such method, sometimes called the method of D-subdivision, makes use of the fact that the roots of Δ depend smoothly on the parameters c, ω, and τ; hence, in the c-ω parameter space, the curves corresponding to the purely imaginary roots are drawn, and then it is determined by implicit differentiation in which direction the roots move when the parameters are

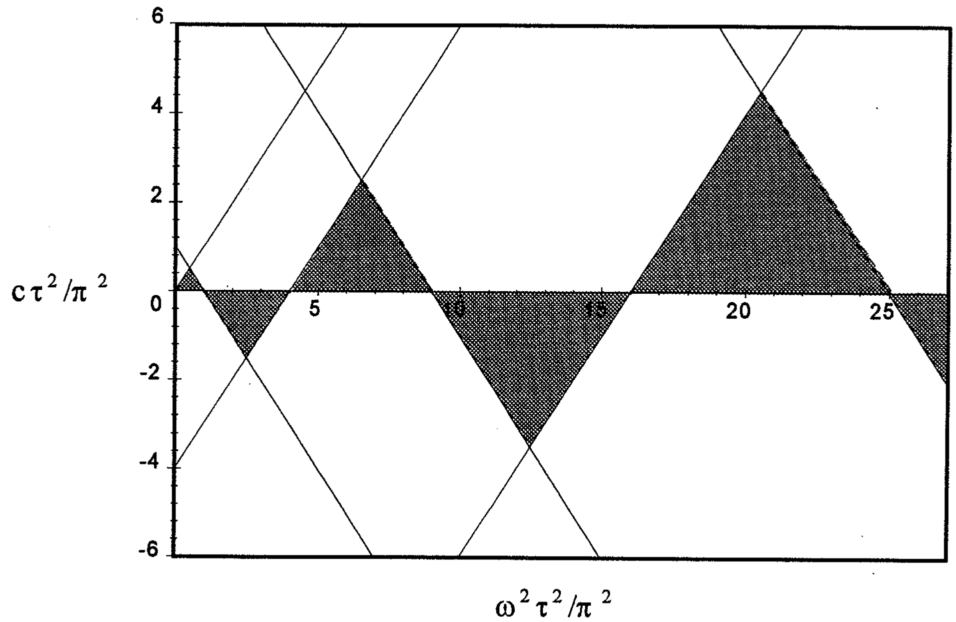

Figure 1: Regions of stability in the parameter space.

changed. This results in partitioning the parameter space into disjoint regions; in each region the number of roots with positive real parts is constant. For details, the reader is referred to Kolmanovskii and Myshkis (1992). The result of applying this procedure to the equation (2) is depicted in Figure 1. Here the boundaries between regions are straight lines. Within the shaded triangular region all roots have negative real parts. The bases of the triangles are located at distances n^2, $n = 0, 1, 2, \ldots$, along the horizontal axis; hence, the conclusion of the theorem follows. ■

Figure 1 displays how the feedback gain c should be chosen to obtain stability. All the lines in the figure have slopes ± 1 and they intersect the horizontal axis at squares of integers. This information suffices to write down an analytical expression for the bounds on c in terms of ω and τ, but we will not pursue this here due to space limitations. Note that outside the shaded regions there are roots with positive real parts; hence, the figure gives sharp information regarding the stability of equation (1). Also note that the sign of the feedback constant c depends on the values of ω and τ. Thus one may need to apply *positive* as well as negative feedback depending on the situation.

It is interesting to note that it is impossible to obtain asymptotic stability when there is no delay in (1). Hence, the presence of delay is in fact necessary for stabilizing the system through position feedback. A related result is given in Atay, this time not for damping out but for modifying the oscillations in a nonlinear system. The practical application of these results to, say, large-scale socio-economic systems of course requires reliable estimates of the natural frequency of oscillations and the amount of delay, but for almost all pairs of the values of these quantities, the delayed feedback accomplishes what cannot be done with instantaneous feedback.

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